Real Analysis 1, Assignment 6, due Oct 26

1. The space ℓ^p

Let $X = \mathbb{N}$, μ the counting measure, and $\ell^p := L^p(X, \mathcal{P}(X), \mu)$, $1 \le p \le \infty$.

i) Identify ℓ^p with a space of real valued sequences. Find an explicit expression for the norm of a sequence x_n. Also formulate the Hölder inequality in terms of sequences.

ii) Show that $\ell^p \subsetneq \ell^q$ for $1 \le p < q \le \infty$.

iii) Show that the sequences $e^{(m)} = \delta_{m,n}$ span a dense subspace of ℓ^p for $1 \le p < \infty$.

2. Generalized Hölder inequality (Folland Exer 6.31)

Suppose that $1 \le p_j \le \infty$ and $\frac{1}{p_1} + \dots \frac{1}{p_n} = \frac{1}{r} \le 1$. Show that if $f_j \in L^{p_j}$ for $j = 1, \dots, n$, then $f_1 \cdots f_n \in L^r$ and

$$||f_1 \cdots f_n||_r \le ||f_1||_{p_1} \cdots ||f_n||_{p_n} .$$

3. L^∞ (Folland Exer 6.2, 6.7)

Prove that L^∞ is a Banach space and that the Hölder inequality holds for $p = \infty$, $p' = 1$. Furthermore, show that if $f \in L^{p_0} \cap L^\infty$ for some $p_0 < \infty$, then $||f||_\infty = \lim_{p \to \infty} ||f||_p$.

4. Weak convergence vs strong convergence (Folland Exer 6.22, 5.63)

Consider $X = [0, 2\pi]$ with the Lebesgue measure. Prove that $f_n(t) = \sin(nt)$ converges weakly to 0 in L^2, but that f_n does not converge strongly in L^2. (Hint: One approach is to prove that in every infinite dimensional Hilbert space, every orthonormal sequence converges weakly to zero.)

5. Characterization of L^p via distribution function (Folland Exer 6.38)

Prove that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{kp} \lambda_f(2^k) < \infty$.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \dots, 6\}$.