# Real Analysis 1, Assignment 7, due Nov 9

## 1. Continuous maps with values in Hausdorff space (c.f. Folland Exer 4.16)

Let $X$ be a topological space, $Y$ a Hausdorff space, and $f, g : X \to Y$ continuous maps.

(a) Prove that $\{x \in X \mid f(x) = g(x)\}$ is closed.

(b) Prove that if $f = g$ on a dense subset of $X$, then $f = g$ on all of $X$.

## 2. Tietze extension theorem for LCH spaces (c.f. Folland Exer 4.46)

Let $X$ be a LCH space. Suppose $K \subset X$ is compact and $U \subset X$ is an open set with $K \subset U$. Prove that if $f : K \to \mathbb{R}$ is continuous, then there exists $F \in C_c(X)$ with $F|_K = f$ and $\mathrm{spt}(F) \subset U$. (Hint: Use local compactness and the Tietze extension theorem for normal spaces)

## 3. Support of a Radon measure (Folland Exer 7.1)

Let $X$ be a LCH space, and $\mu$ a Radon measure.

(a) Let $N_\mu$ be the union of all open $U \subset X$ such that $\mu(U) = 0$. The support of $\mu$ is defined to be the set $\mathrm{spt}(\mu) := X \setminus N_\mu$. Prove that $\mathrm{spt}(\mu)$ is closed and that $\mu(X \setminus \mathrm{spt}(\mu)) = 0$.

(b) Prove that $x \in \mathrm{spt}(\mu)$ if and only if $\int_X f\, d\mu > 0$ for every $f \in C_c(X, [0, 1])$ with $f(x) > 0$.

## 4. Measure induced by $L^1$-function (Folland Exer 7.8)

Let $X$ be a LCH space, and $\mu$ a Radon measure. Prove that if $f : X \to [0, \infty)$ is a nonnegative integrable function, then $\nu(E) = \int_E f\, d\mu$ is a Radon measure.

## 5. An application of Lusin's theorem

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue-measurable function with $f(x + y) = f(x) + f(y)$.

(a) Prove that $f$ is continuous. (Hint: Use Lusin's theorem to show that $f$ is continuous at the origin.)

(b) Show that $f(x) = cx$, where $c = f(1)$.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let $s$ be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \dots, 6\}$.