Real Analysis 1, Assignment 8, due Nov 16

1. Concluding integrability (Stein-Shakarchi Exer 2.18)

Assume $f : [0, 1] \to \mathbb{R}$ is a Lebesgue-measurable function such that $\int_0^1 \int_0^1 |f(x) - f(y)| \, dx \, dy < \infty$. Prove that $\int_0^1 |f| \, dx < \infty$.

2. Application of Fubini-Tonelli (Folland Exer 2.57)

Show that $\int_0^\infty e^{-sx} \frac{\sin x}{x} \, dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy} \sin x$ with respect to x and y. (Hint: $\tan(\frac{\pi}{2} - \theta) = \frac{1}{\tan \theta}$).

3. The σ-finiteness assumption in Fubini-Tonelli (Folland Exer 2.46)

Let $X = Y = [0, 1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, μ = Lebesgue measure, ν = counting measure. Let $D = \{(x, x) | x \in [0, 1]\}$ be the diagonal. Show that $\int\int \chi_D \, d\mu d\nu$, $\int\int \chi_D \, d\nu d\mu$ and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (Hint: To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$ go back to the definition of $\mu \times \nu$).

4. Version of Fubini-Tonelli for complete measures (Folland Exer 2.49)

Prove the version of the Fubini-Tonelli theorem for complete measures (Folland Thm 2.39).

5. Approximation by smooth functions

Fix a smooth nonnegative function $\eta \in C_c^\infty(\mathbb{R}^n)$ with $\int \eta = 1$. For $\varepsilon > 0$ set $\eta_\varepsilon(x) = \frac{1}{\varepsilon^n} \eta\left(\frac{x}{\varepsilon}\right)$. For any $u \in C_c(\mathbb{R}^n)$ consider

$$u_\varepsilon(x) := \int_{\mathbb{R}^n} u(y) \eta_\varepsilon(x - y) \, dy \, .$$

a) Prove that $u_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ for every $\varepsilon > 0$.

b) Prove that $u_\varepsilon \to u$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$ for any $1 \leq p < \infty$.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.