Real Analysis 1, Assignment 9, due Nov 23

1. Total variation (c.f. Folland Exer 3.3, 3.7)

Let ν be a signed measure on a measurable space $(X, \mathcal{M})$. Recall that the total variation is defined by $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ denotes the Jordan decomposition. Prove that

$$|\nu|(E) = \sup\{\sum_{i=1}^{\infty} |\nu(E_i)| : E = \cup_{i=1}^{\infty} E_i, E_i \in \mathcal{M} \text{ disjoint } \}$$
$$= \sup\{\int_E f\, d\nu : f \text{ measurable }, |f| \leq 1\}$$

for all $E \in \mathcal{M}$. Conclude that $||\nu|| := |\nu|(X)$ defines a norm on the space of all finite signed measures on $(X, \mathcal{M})$.

2. Mutually singular measures (Folland Exer 3.2)

Let ν be a signed measure and μ be a positive measure on a measurable space $(X, \mathcal{M})$, and let $E \in \mathcal{M}$. Prove that

(a) E is ν-null $\quad \Leftrightarrow \quad |\nu|(E) = 0$.

(b) $\nu \perp \mu \quad \Leftrightarrow \quad \nu^+ \perp \mu$ and $\nu^- \perp \mu \quad \Leftrightarrow \quad |\nu| \perp \mu$.

3. Absolute continuity of measures (Folland Exer 3.8)

Let ν be a signed measure and μ be a positive measure on a measurable space $(X, \mathcal{M})$. Prove that: $\nu \ll \mu \quad \Leftrightarrow \quad \nu^+ \ll \mu$ and $\nu^- \ll \mu \quad \Leftrightarrow \quad |\nu| \ll \mu$.

4. Radon-Nikodym calculus (cf. Folland page 91, Exer 3.16)

In this question all measures under consideration are σ-finite, and are on some fixed measurable space $(X, \mathcal{M})$. If ν is a signed measure and μ is a positive measure, and $\nu \ll \mu$, then by the Radon-Nikodym theorem there exist a unique (we tacitly identify functions that agree μ-almost everywhere) extended μ-measurable function f such that $\nu(E) = \int_E f\, d\mu$ for all $E \in \mathcal{M}$. We call f the Radon-Nikodym derivative of ν with respect to μ and write $\frac{d\nu}{d\mu} = f$.

(a) Prove that if ν, λ are signed measures, μ is a positive measure, $\nu \ll \mu$, $\lambda \ll \mu$, and $a, b \in \mathbb{R}$, then $a\nu + b\lambda \ll \mu$ and $\frac{d(a\nu+b\lambda)}{d\mu} = a\frac{d\nu}{d\mu} + b\frac{d\lambda}{d\mu}$.

(b) Suppose μ, ν are positive measures with $\nu \ll \mu$, and let $\lambda := \mu + \nu$. Let $f := \frac{d\nu}{d\lambda}$. Prove that $0 \leq f \leq 1$ (μ-a.e.) and that $\frac{d\nu}{d\mu} = f/(1-f)$.

5. Convergence of measures (c.f. Folland Exer 3.24)

Let $\mathcal{M}(\mathbb{R}) := \{\mu : \mathcal{B}_{\mathbb{R}} \to \mathbb{R} \,|\, \mu \text{ signed Radon measure}\}$ be the space of finite signed Radon (or equivalently Borel) measures on $\mathbb{R}$. Let $\mu_n, \mu \in \mathcal{M}(\mathbb{R})$. We say μ_n converges to μ vaguely if $\int_{\mathbb{R}} f d\mu_n \to \int_{\mathbb{R}} f d\mu$ for all $f \in C_0(X)$, and μ_n converges to μ in total variation if $||\mu_n - \mu|| \to 0$.

(a) Prove that convergence in total variation implies vague convergence, but that the converse implication doesn't hold.

(b) Explain why vague convergence is sometimes called weak-star convergence.

(c) Give an example of a sequence $\mu_n \in \mathcal{M}(\mathbb{R})$ such that μ_n converges to 0 vaguely, but such that $\int_{\mathbb{R}} f d\mu_n \nrightarrow \int_{\mathbb{R}} f d\mu$ for some bounded measurable f with compact support.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.