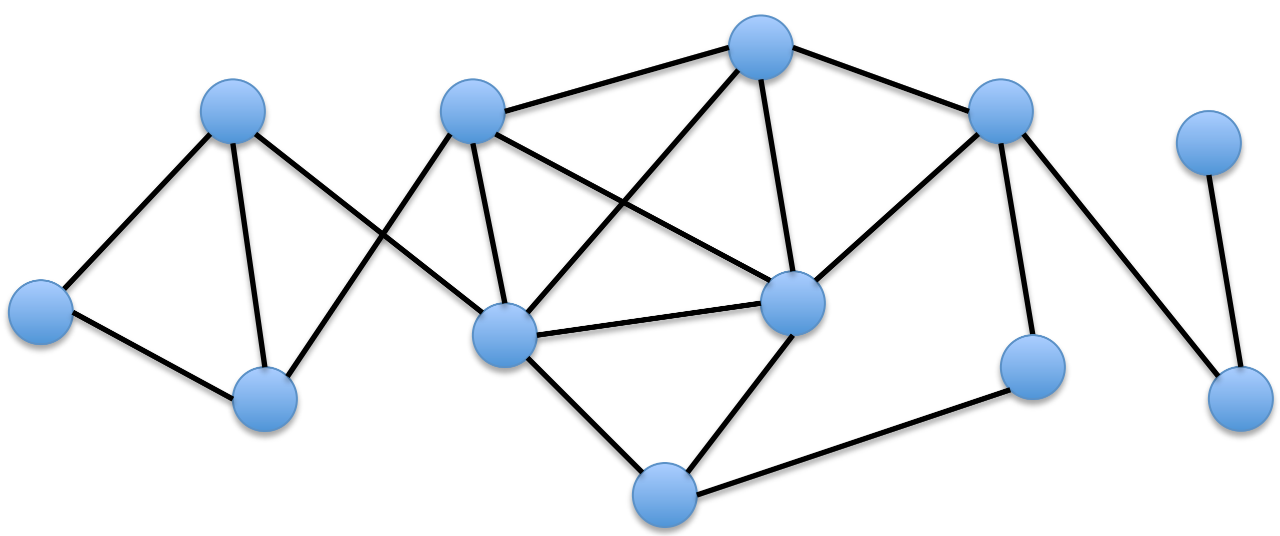
Ehrenfeucht-Fraïssé Games

Relational Structures

- We will consider language consisting of ***relation symbols*** only (no constant or function symbols).
- A **relational structure** $\mathbf{A} = (A, R_1, \ldots, R_t)$ consists of
  – a set A (called the universe of $\mathbf{A}$)
  – a sequence of relations $R_i \subseteq A^{r_i}$
- $\mathbf{A}$ and $\mathbf{B}$ are structures in the same relational language.

Graphs

- A **graph** $G = (V, \sim)$ consists of
  - a set V of vertices
  - a binary relation $\sim \subseteq V^2$ (anti-reflexive and symmetric)



First-Order Logic on Graphs

- **First-order formulas** are built from:

  atomic formulas

  $$x = y \qquad x \sim y$$

  via ***connectives***

  $$\neg\varphi \qquad \varphi \wedge \psi \qquad \varphi \vee \psi \qquad \varphi \rightarrow \psi$$

  and ***quantifiers***

  $$\forall x \;\varphi(x) \qquad \exists x \;\varphi(x)$$

First-O

adjacency relation

“there is an edge between x and y”

- **First-order formulas** …
  atomic formulas

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Variables range over ***vertices***

Definable and Axiomatizable Properties

Definability and Axiomatizability

- Let $\mathcal{C}$ be a class of graphs (i.e. a *graph property*)
- $\mathcal{C}$ is **FO-definable** if there is a single sentence φ such that

$$G \models \varphi \quad \Leftrightarrow \quad G \in \mathcal{C}$$

- $\mathcal{C}$ is **FO-axiomatizable** if there is a set of sentences Σ such that

$$G \models \Sigma \quad \Leftrightarrow \quad G \in \mathcal{C}$$

FO-Definable Graph Properties

- “no isolated vertex” (i.e. every vertex has a neighbor)

$$\forall x \exists y \; x \sim y$$

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$$\forall x \exists y \ x \sim y$$

- "diameter is ≤ 2"

$$\forall x \forall y \big[x = y \vee x \sim y \vee \exists z (x \sim z \wedge z \sim y) \big]$$

FO-Definable Graph Properties

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FO-Definable Graph Properties

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$$\forall x \exists y \; x \sim y$$

- “diameter is ≤ 2”

$$\forall x \forall y \left[x = y \vee x \sim y \vee \exists z (x \sim z \wedge z \sim y) \right]$$

- “3-regular” (every vertex has exact 3 neighbors)
- “girth > 17” (no induced cycle of length ≤ 17)

FO Definable Properties of Graphs

- But not every property of graphs is FO definable.
- For example, **3-colorability** is naturally expressed in ***second-order logic*** (allowing quantification over sets and relations). But cannot be expressed in first-order logic.

$$\exists R \exists B \exists G \Big(\forall x\, R(x) \vee B(x) \vee G(x) \Big) \wedge$$

$$\forall x \forall y \;\; Edge(x,y) \rightarrow \neg \left(\begin{array}{l} (R(x) \wedge R(y)) \vee \\ (B(x) \wedge B(y)) \vee \\ (G(x) \wedge G(y)) \end{array} \right)$$

FO Definable Properties of Graphs

- To show that a class $\mathcal{C}$ is first-order definable: simply write down a first-order formula that defines it.
- How can we show that a class $\mathcal{C}$ is *not* first-order definable?

Quantifier-Rank & k-Equivalence

Quantifier-rank

- **Quantifier-rank** of a formula is the maximum nesting depth of quantifiers

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- Rank-3 formula:

$$\exists x\,[\forall y\,[E(x,y) \vee \exists z\,[E(y,z) \wedge \neg(x = z)]]]$$

Quantifier-rank

- **Quantifier-rank** of a formula is the maximum nesting depth of quantifiers

- Inductive definition:

  $\text{rank}(x = y) = \text{rank}(R(x_1,\ldots,x_r)) = 0,$

  $\text{rank}(\neg\varphi) = \text{rank}(\varphi),$

  $\text{rank}(\varphi \wedge \psi) = \text{rank}(\varphi \vee \psi) = \max\{\text{rank}(\varphi), \text{rank}(\psi)\},$

  $\text{rank}(\forall x\ \varphi(x)) = \text{rank}(\exists x\ \varphi(x)) = \text{qr}(\varphi) + 1$

k-Equivalence

- Structures **A** and **B** are **k-equivalent** (denoted **A** $\equiv_k$ **B**) if they satisfy the same sentences of quantifier-rank k.
- In other words, **A** $\equiv_k$ **B** iff **A** and **B** cannot be distinguished by any first-order sentence of quantifier-rank k.

The Ehrenfeucht-Fraïssé Game

EF Game

- The **k-round Ehrenfeucht-Fraisse game** on structures **A** and **B** has two players, ***Spoiler*** and ***Duplicator***

EF Game

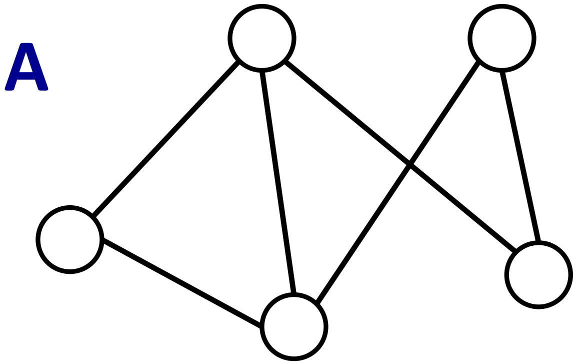
- The **k-round Ehrenfeucht-Fraisse game** on structures **A** and **B** has two players, ***Spoiler*** and ***Duplicator***
- The game captures the *quantifier-rank* needed to distinguish **A** and **B** in first-order logic

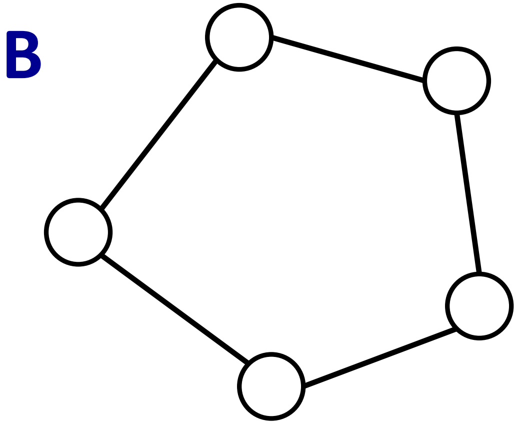
Duplicator's goal: prove $\mathbf{A} \equiv_k \mathbf{B}$

Spoiler's goal: refute $\mathbf{A} \equiv_k \mathbf{B}$

EF Game

- <u>In each of k rounds:</u>
  1. ***Spoiler*** picks an element in either **A** or **B**,
  2. ***Duplicator*** picks an element in the other structure





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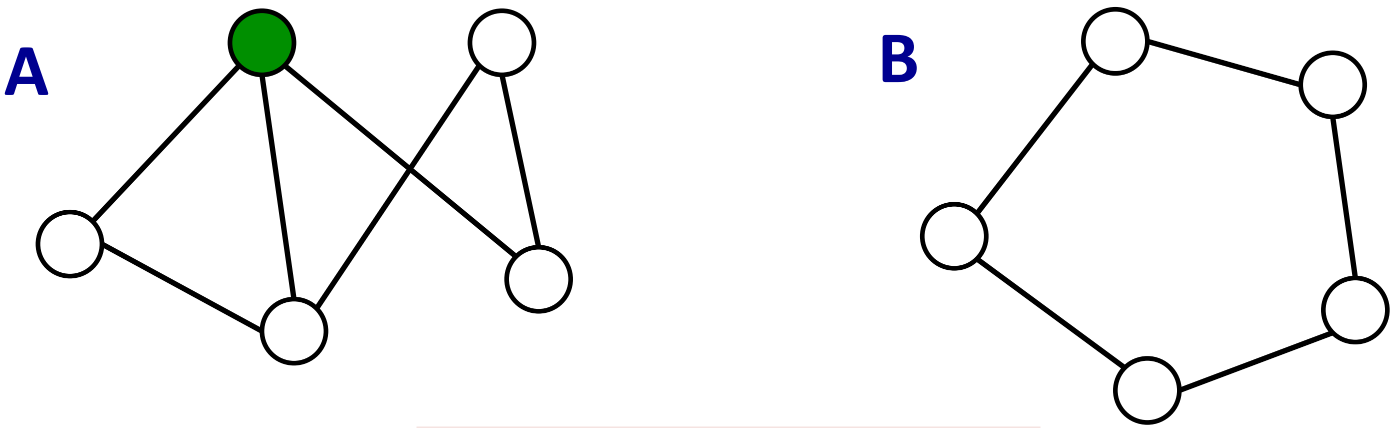

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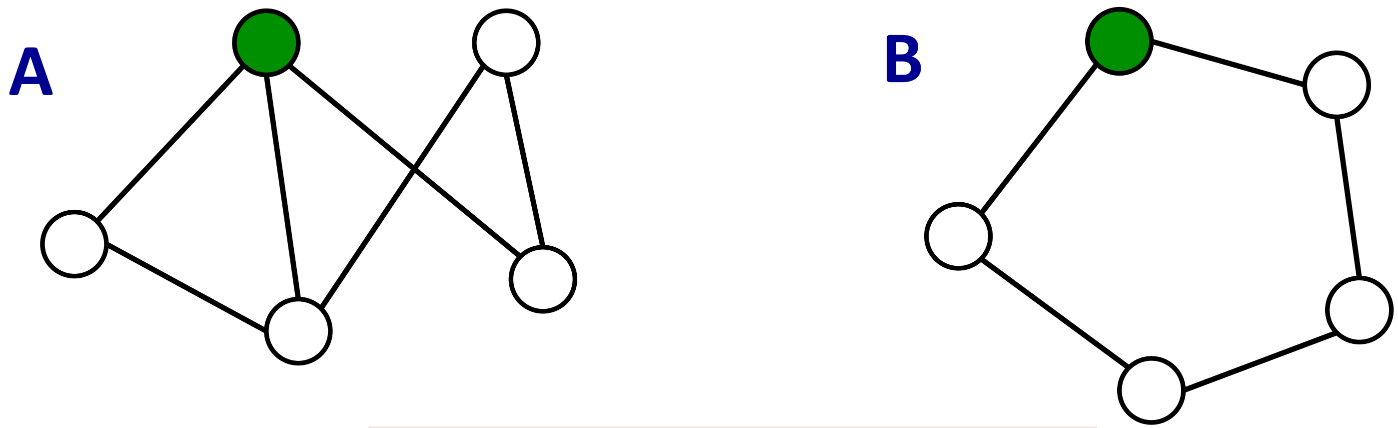

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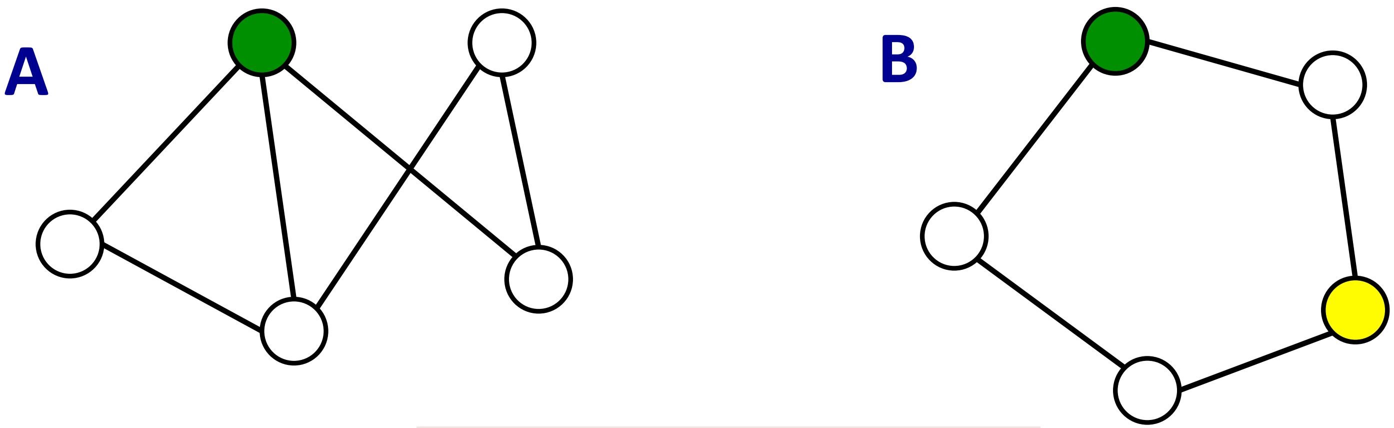

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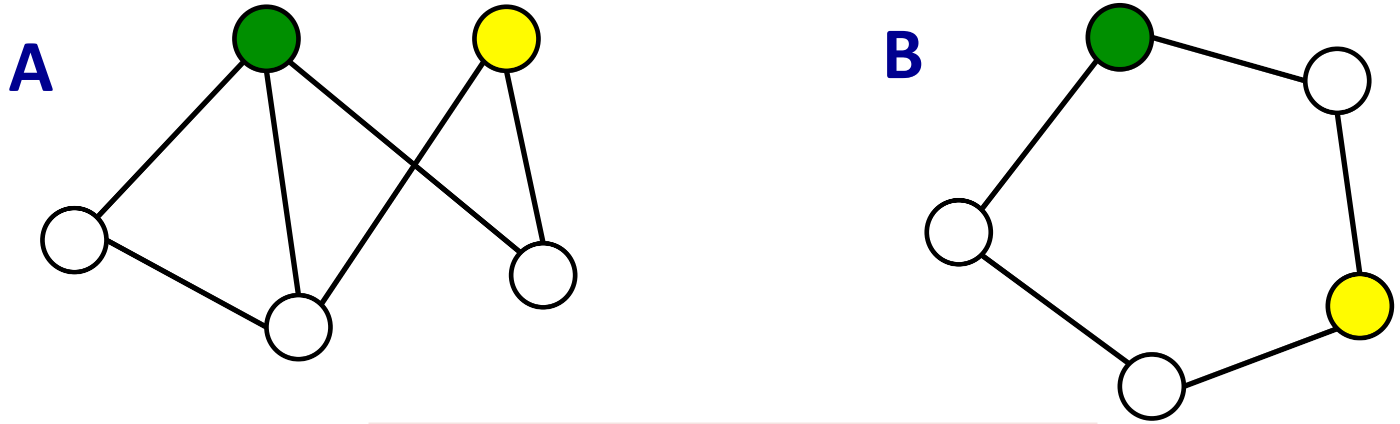

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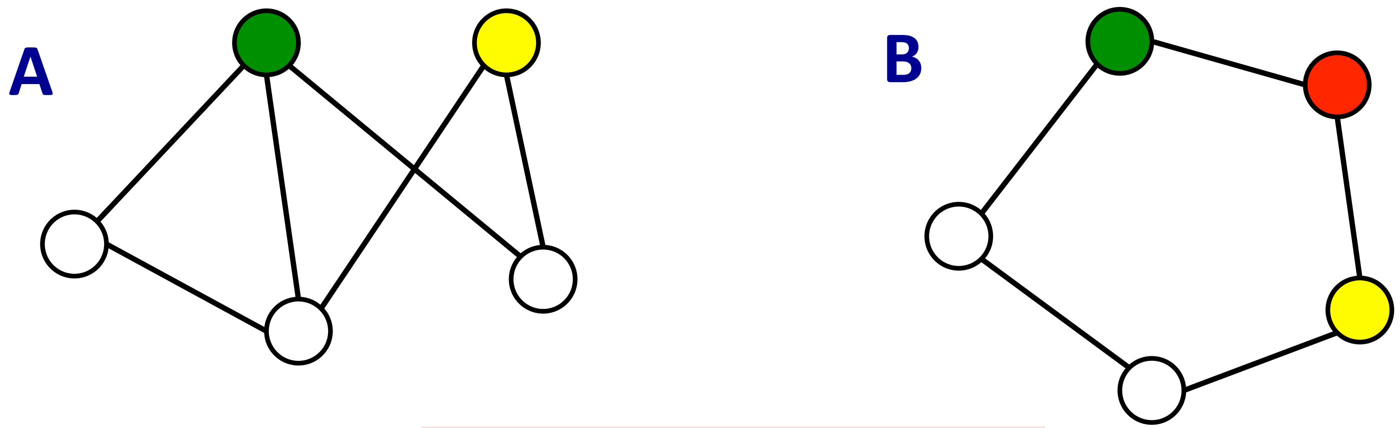

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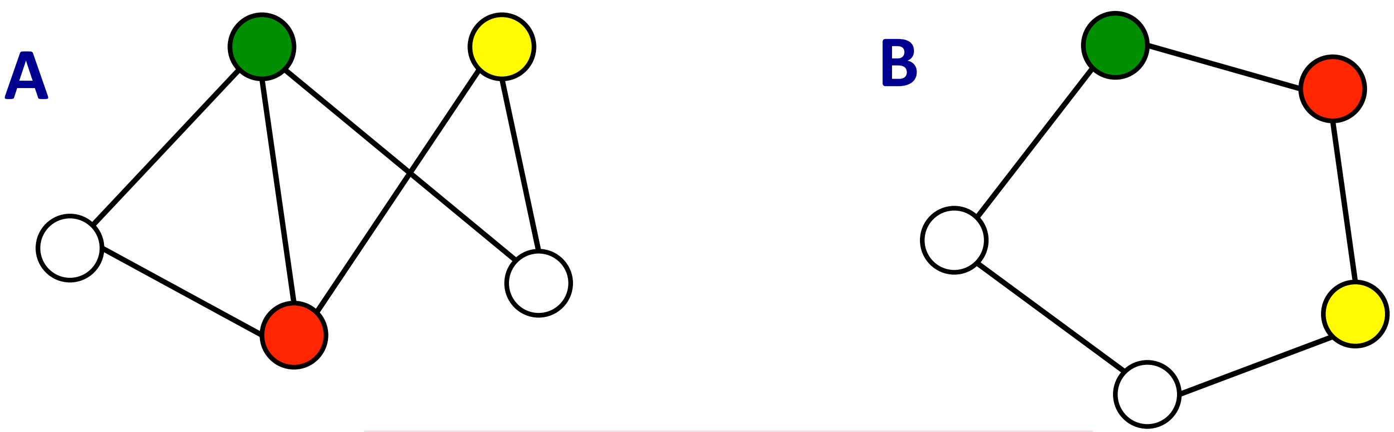
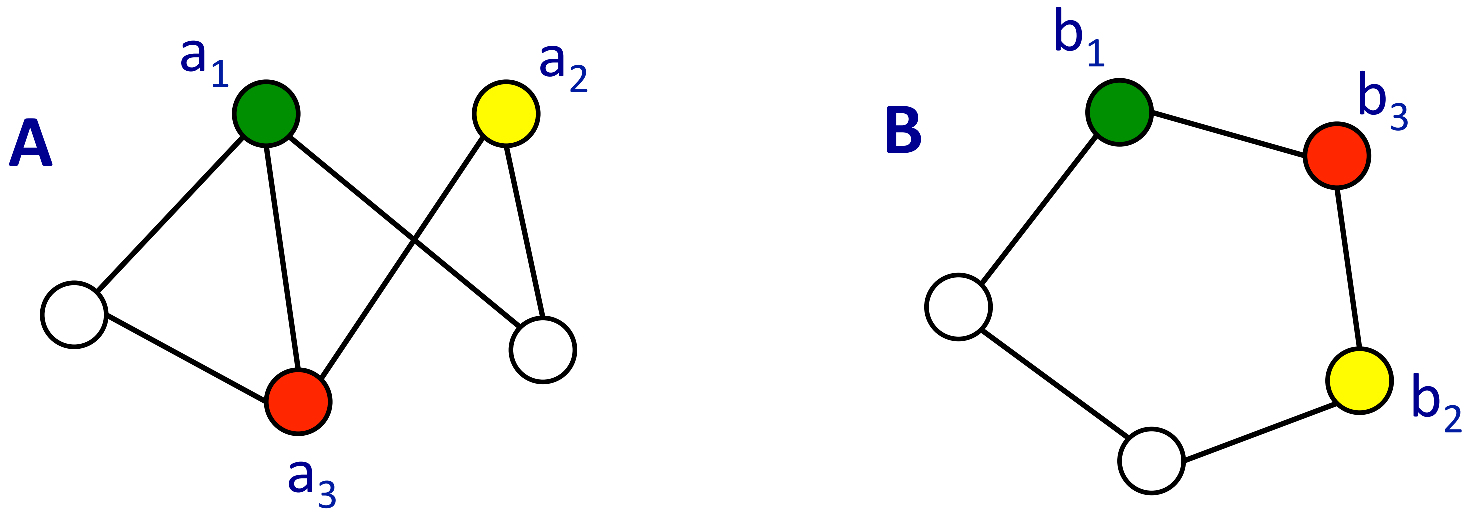


Round 3: *Duplicator*

EF Game

- After k rounds: There are distinguished elements $a_1, \ldots, a_k$ in **A** and $b_1, \ldots, b_k$ in **B**



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$$\{a_1 \mapsto b_1, \ldots, a_k \mapsto b_k\}$$

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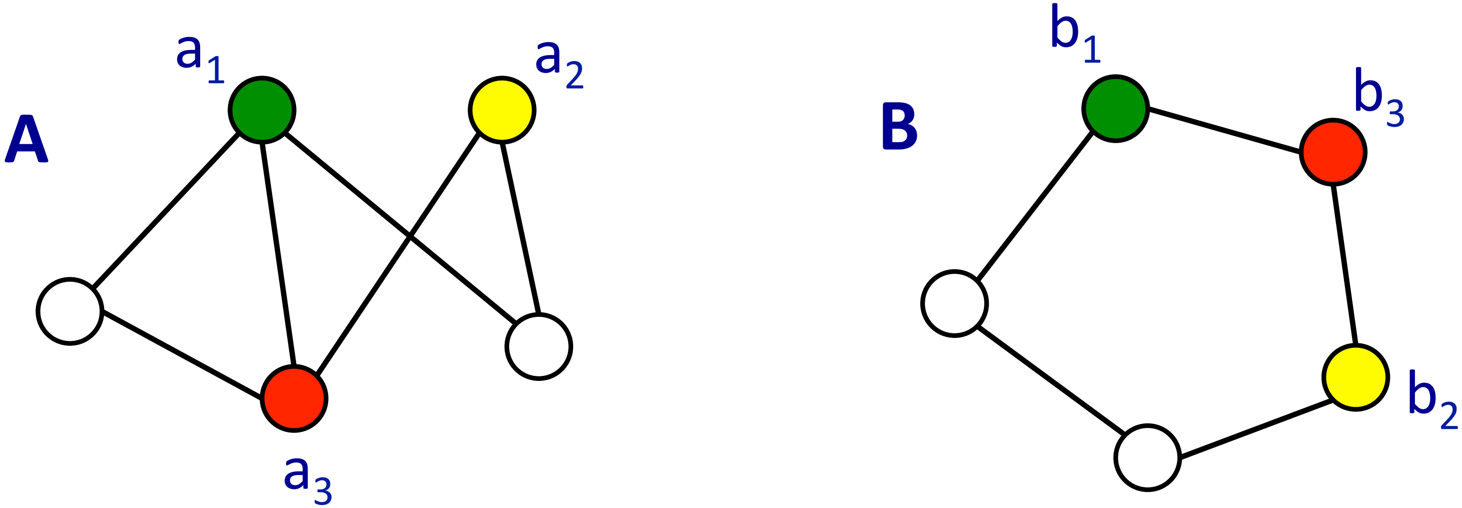

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If **A** and **B** are graphs, ***partial isomorphism*** means:

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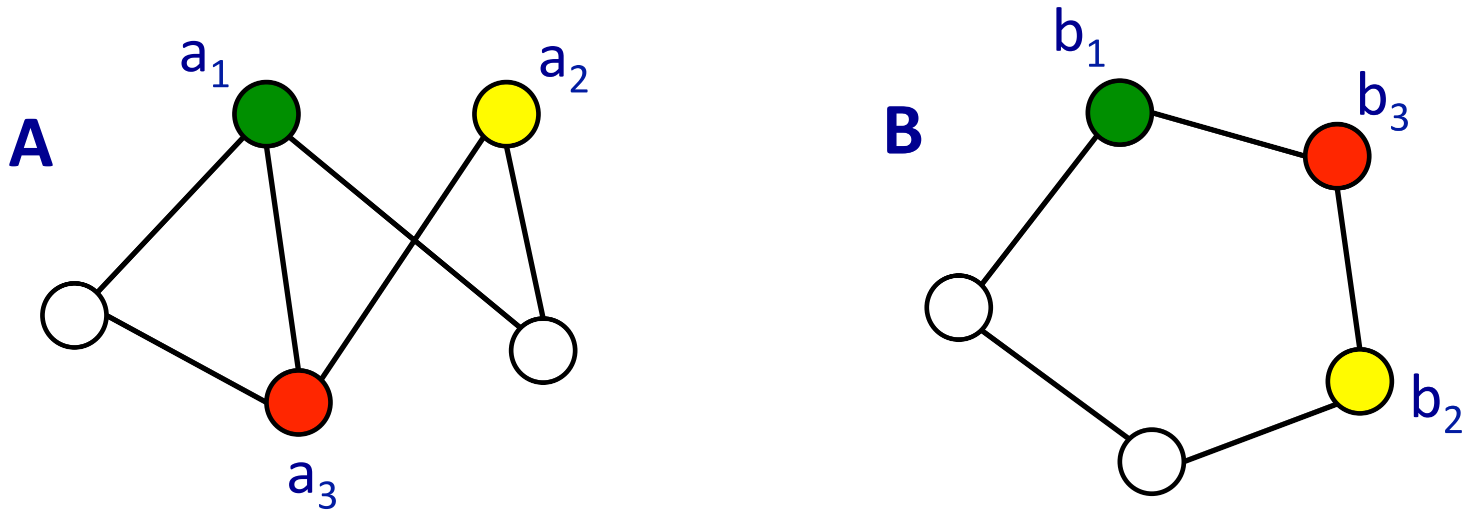

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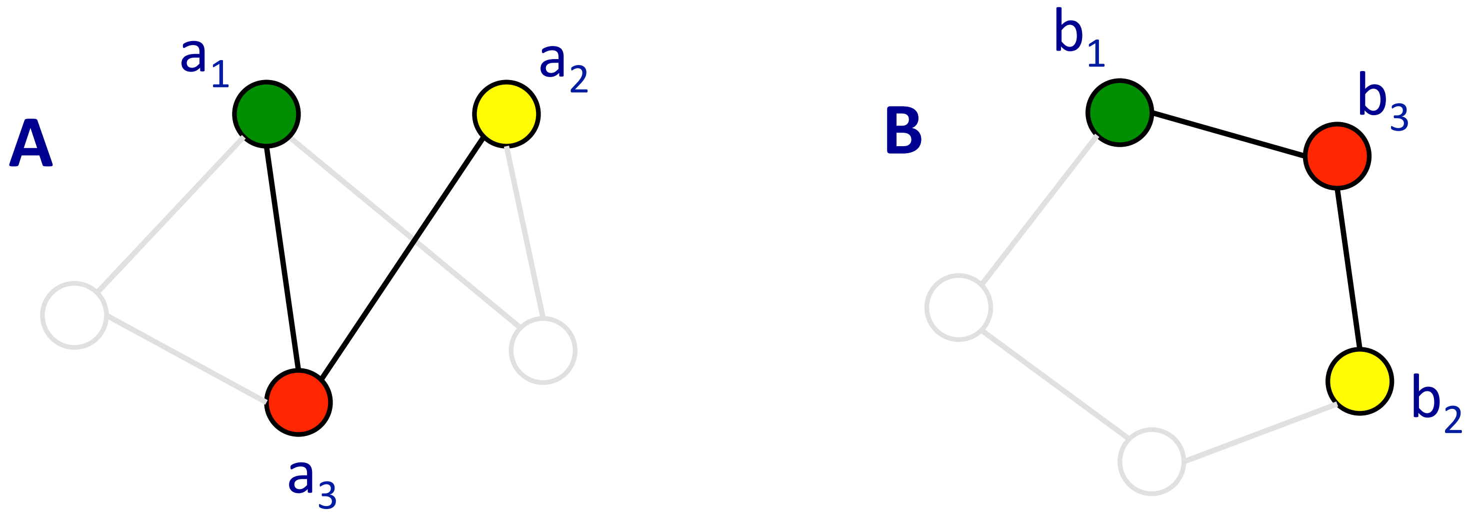

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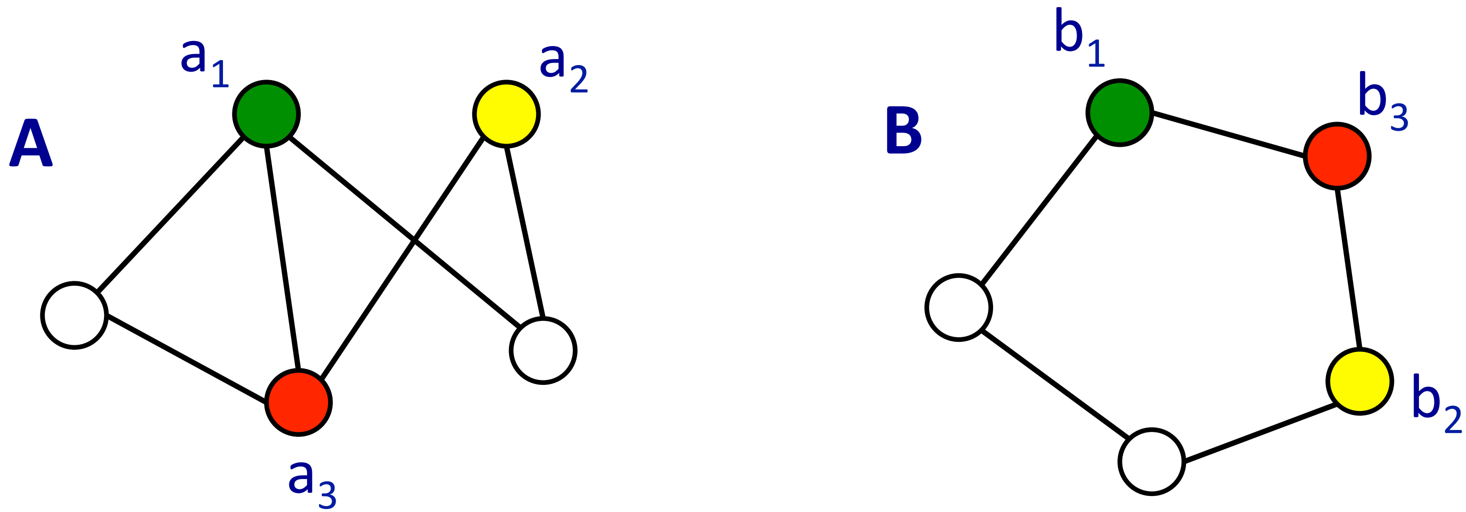

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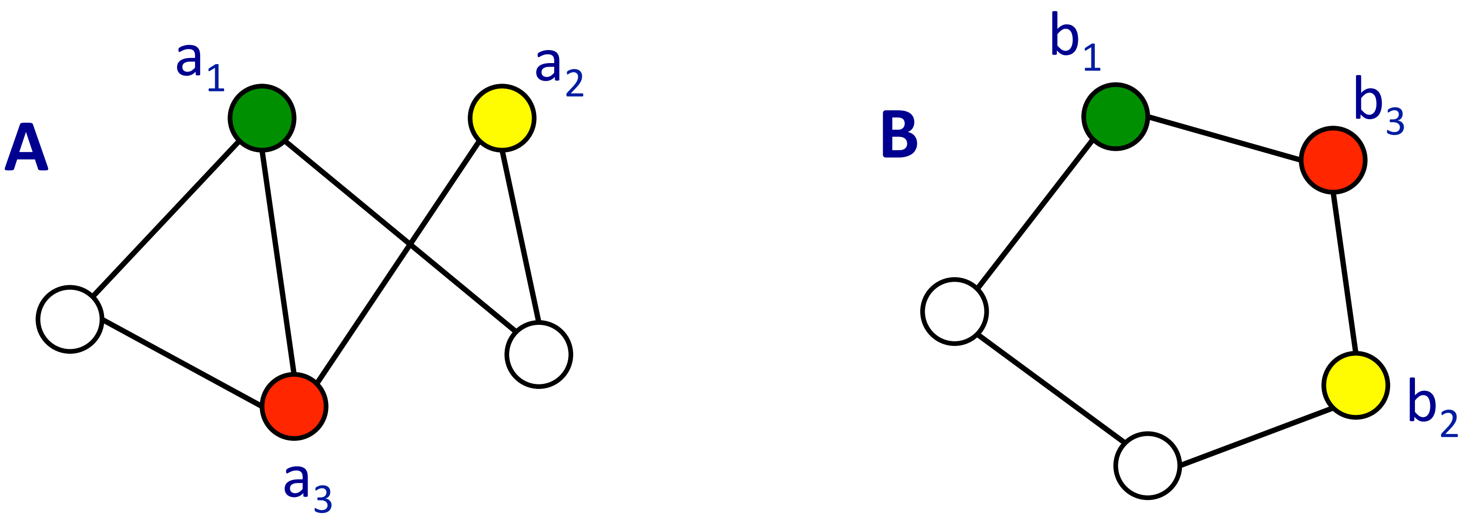

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- Fact: There exists a ***winning strategy*** for one of the two players.

EF Game

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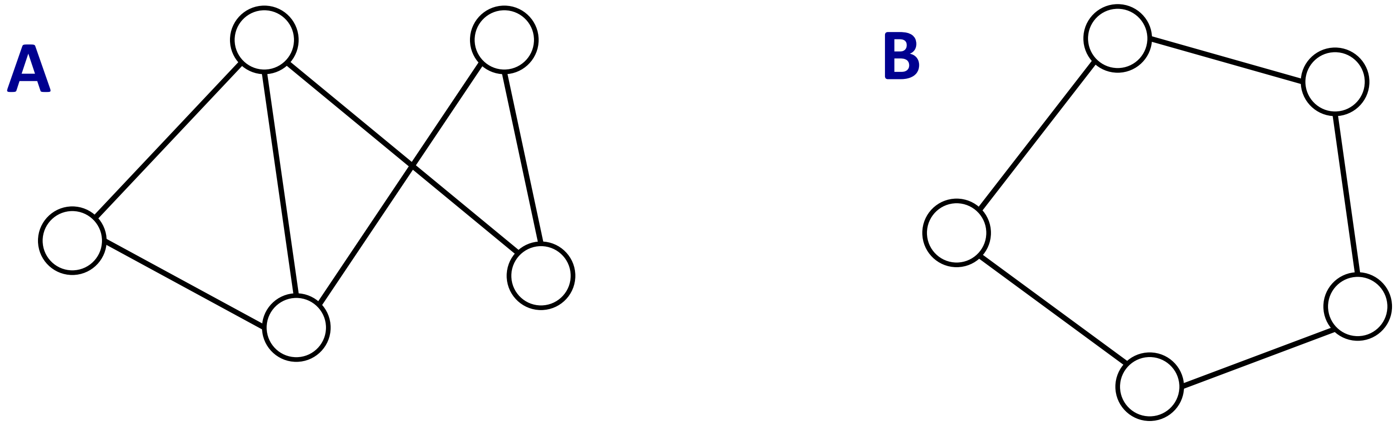
> True of any deterministic zero-sum game of finite length.

- ***Spo*** $b_k\}$ is not a partial i morphism)

- <u>Fact</u>: There exists a ***winning strategy*** for one of the two players.

- In our example, Spoiler has a winning strategy in the 3-round game:



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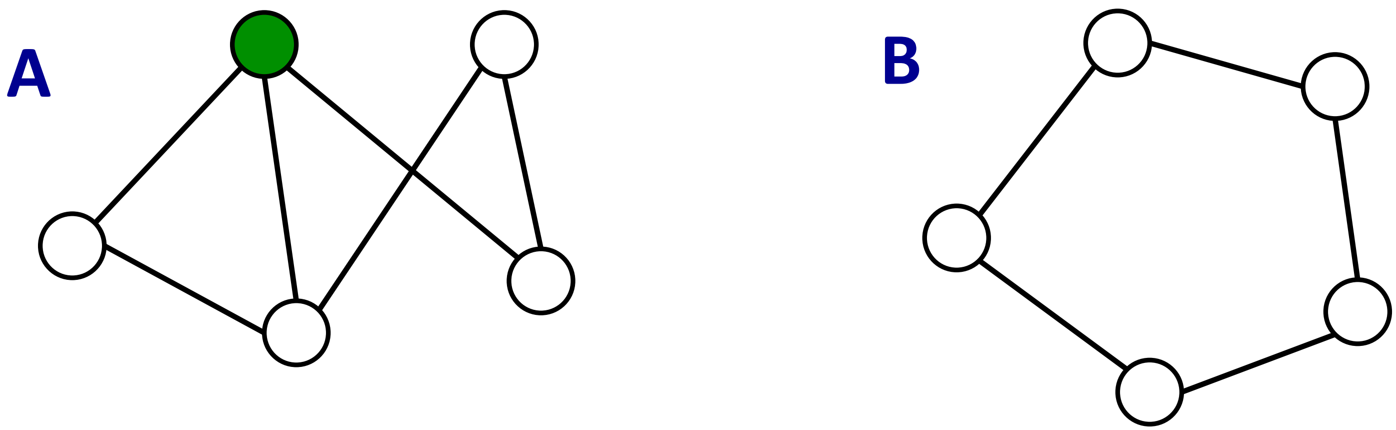

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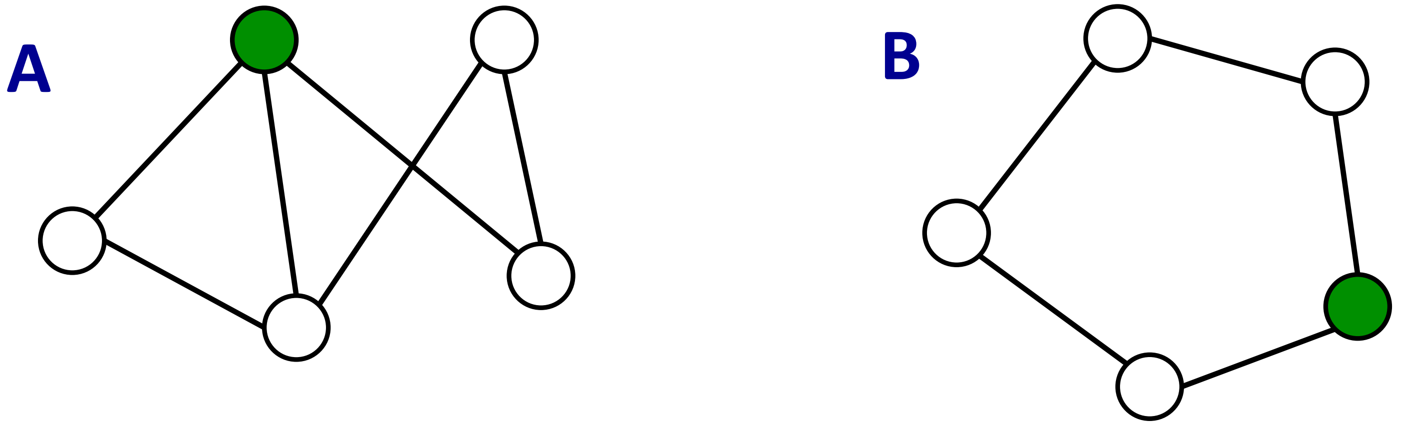

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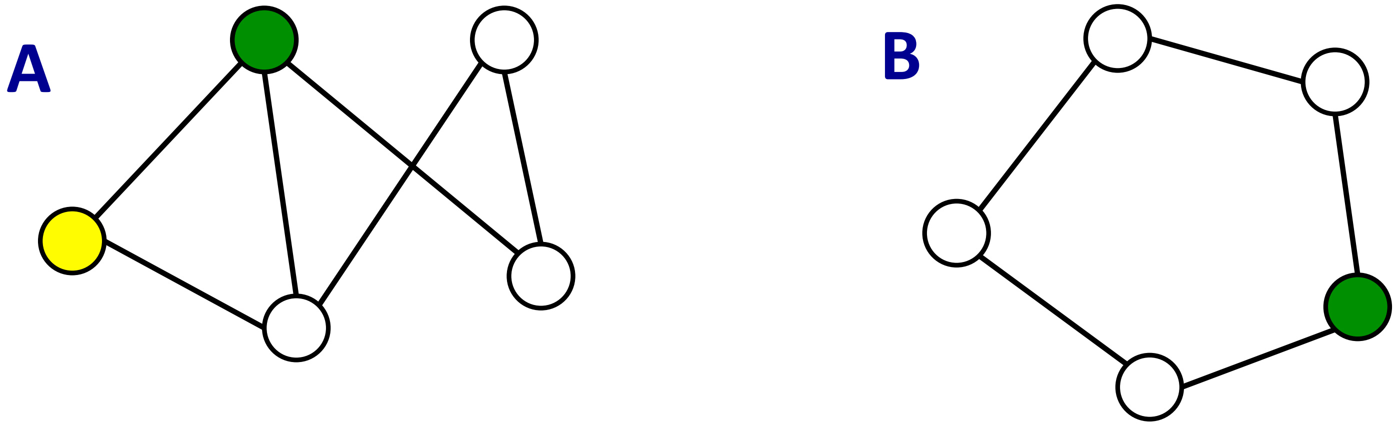

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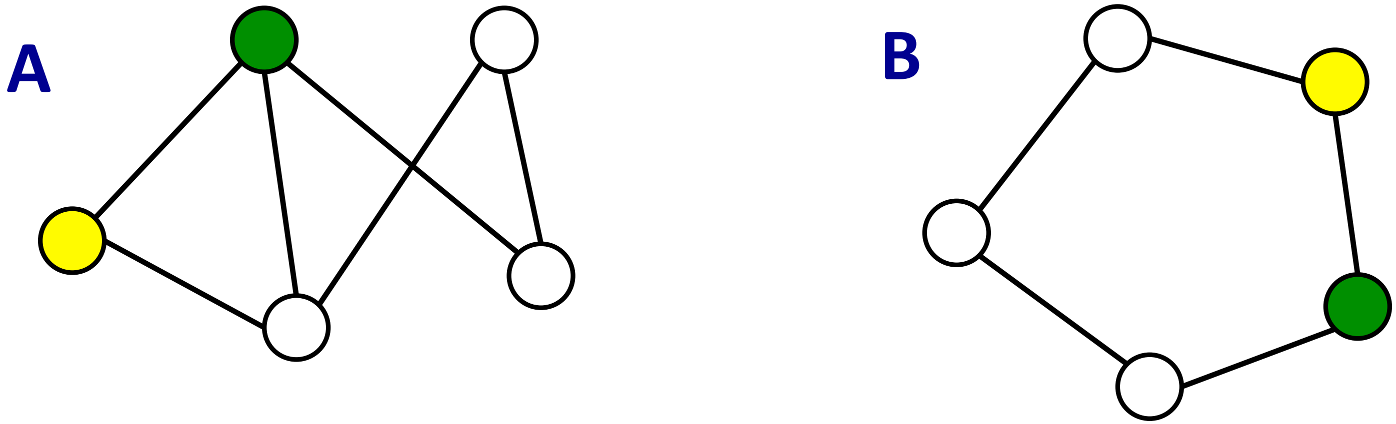

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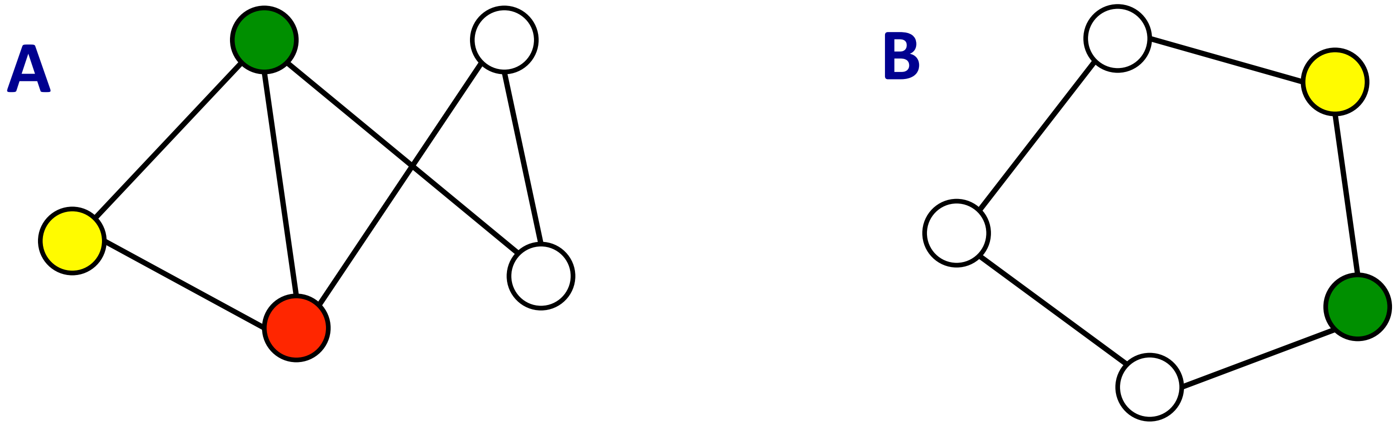

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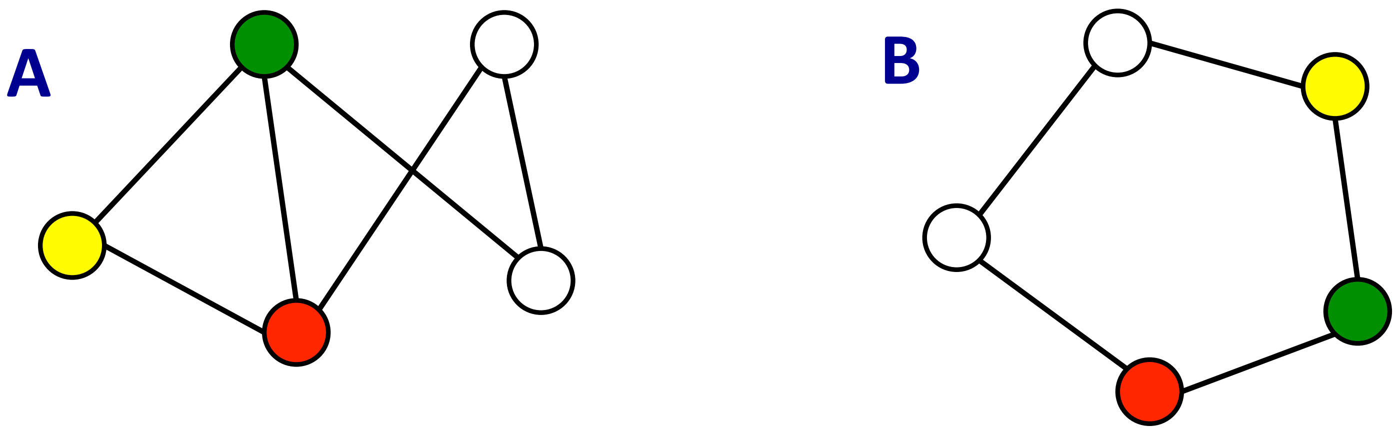

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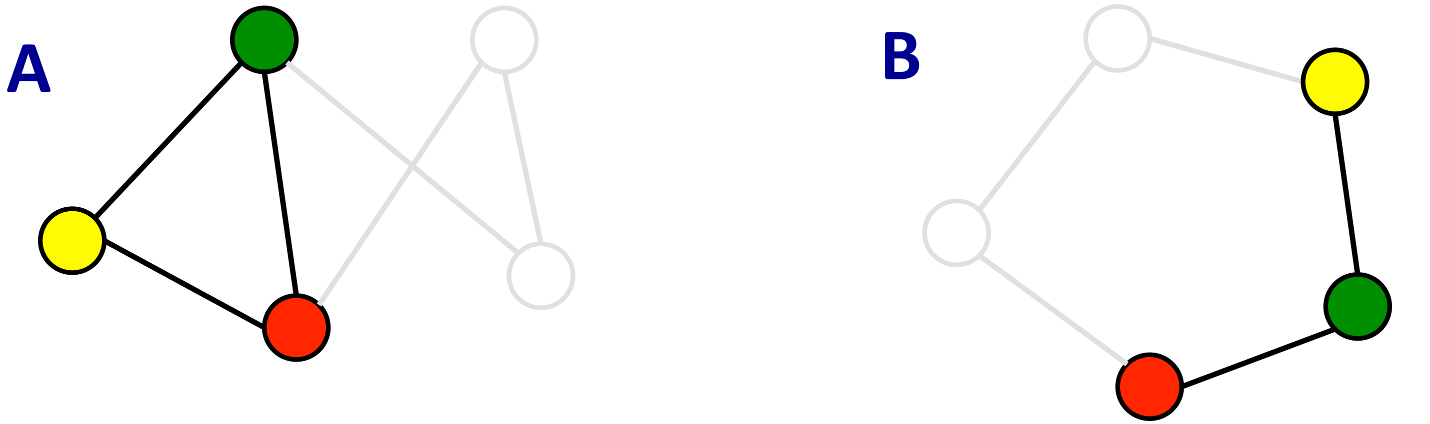

<u>not</u> a partial isomorphism: ***Spoiler*** wins

Fundamental Theorem of EF Games

Theorem

Duplicator has a winning strategy in the k-round EF game on **A** and **B** if, and only if, **A** $\equiv_k$ **B**.

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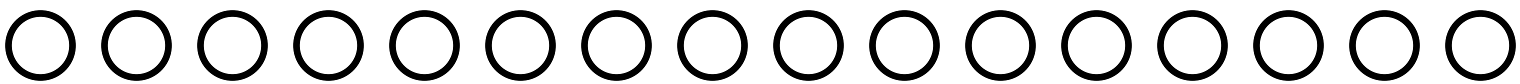
- Proof by induction on k.

Repertoire of Winning Strategies

Linear orders

- Finite structures **A** with universe {1, …, n} and binary relation $<$



- We say **A** is EVEN if n is even

Linear orders

<u>Theorem</u>

The class of EVEN linear orders is *not* FO definable.

Linear orders

Theorem

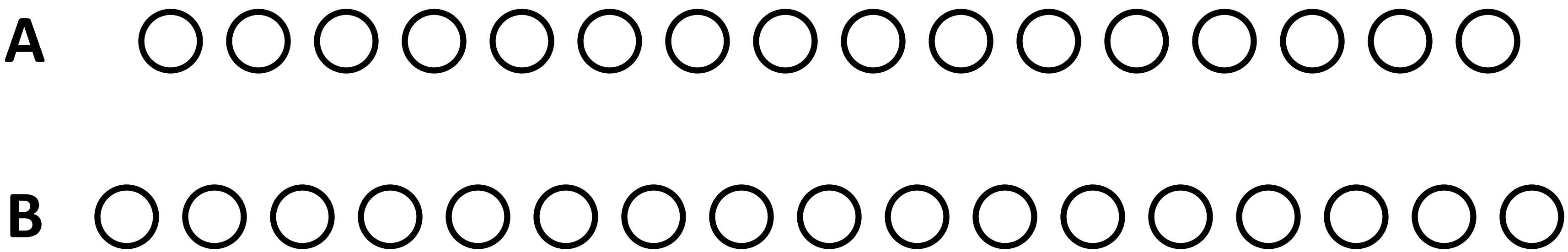
The class of EVEN linear orders is *not* FO definable.

Proof

For arbitrary k, we show that **A** $\equiv_k$ **B** where **A** is a linear order of even size 2^k and **B** is a linear order of odd size $2^k + 1$.

We prove **A** $\equiv_k$ **B** by giving a winning strategy for ***Duplicator*** in the k-round EF game on **A** and **B**.

Winning strategy (by picture)



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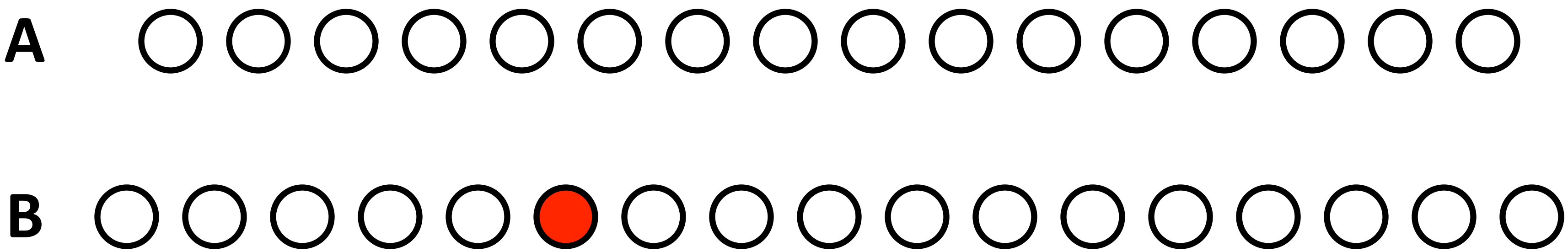

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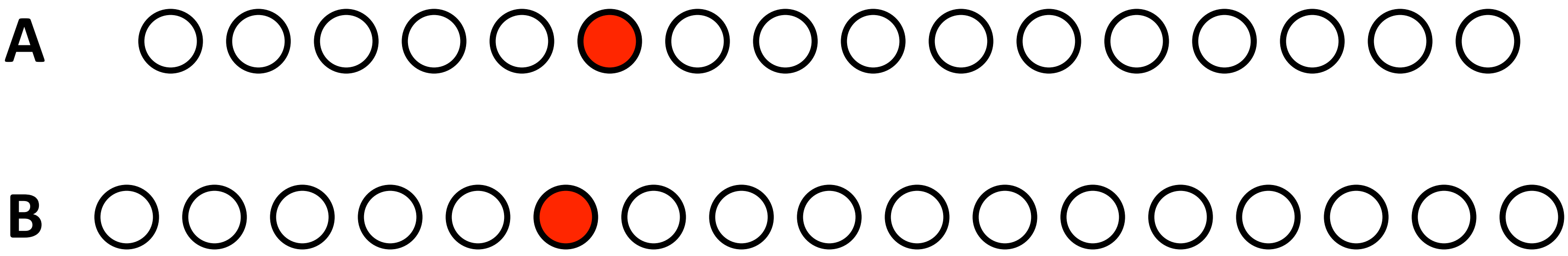

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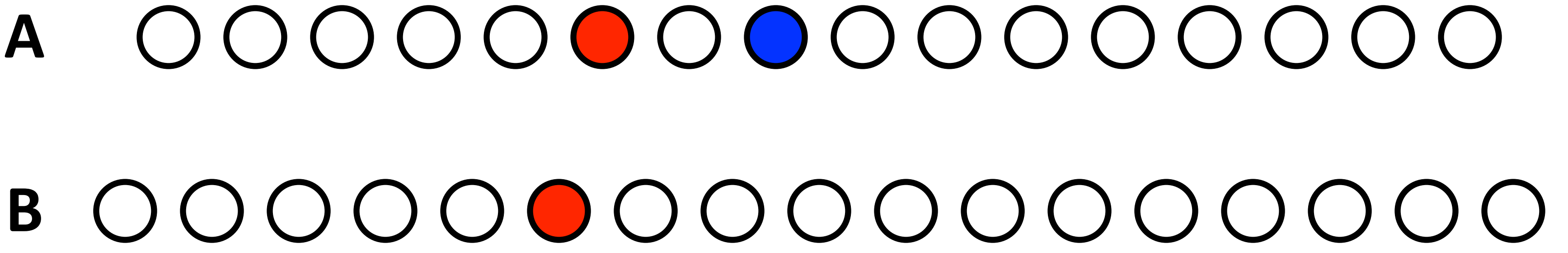
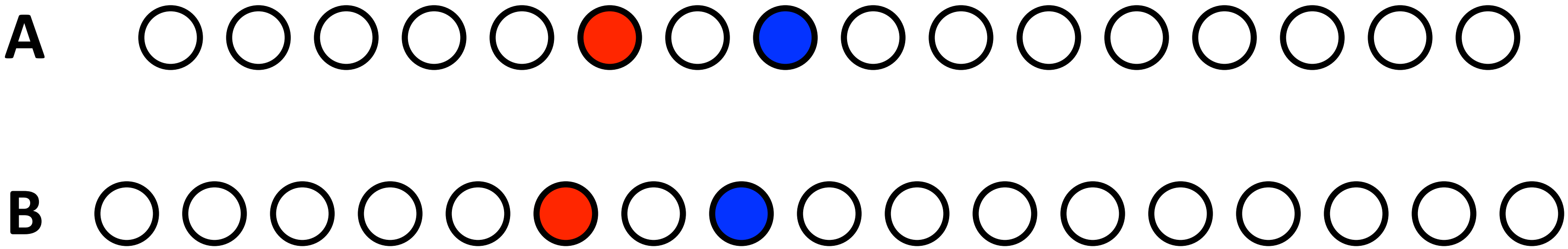


Round 2: ***Spoiler***

Winning strategy (by picture)



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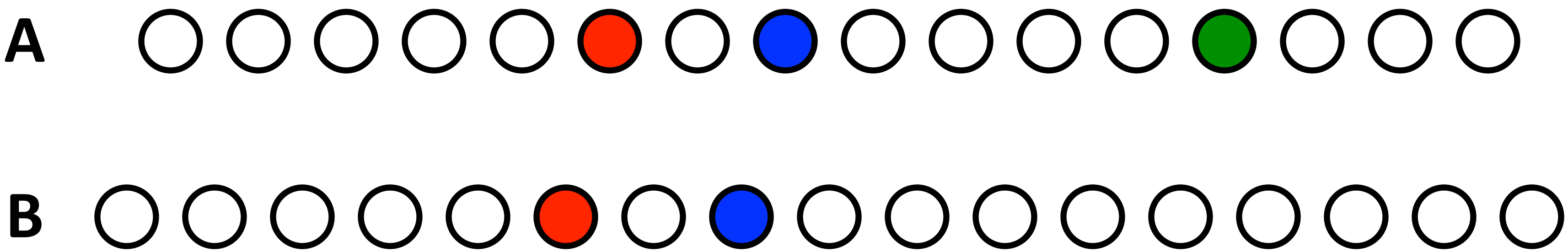

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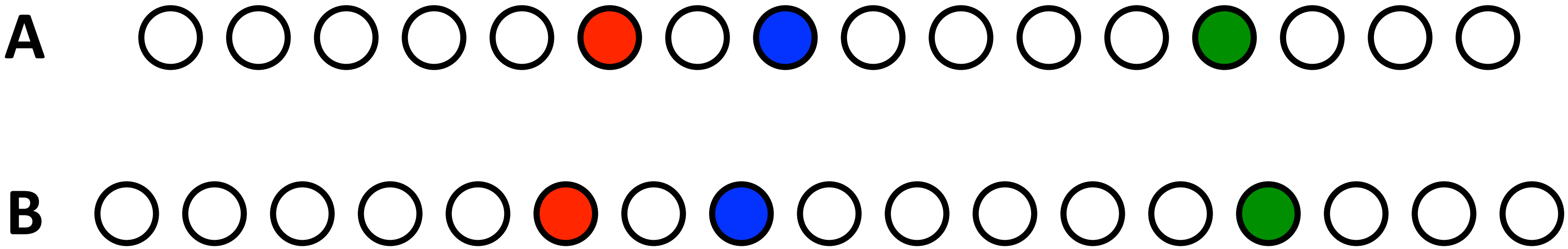

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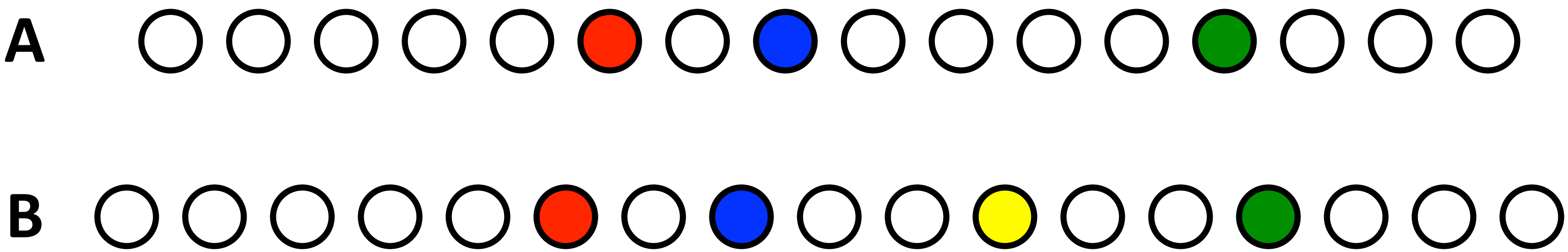
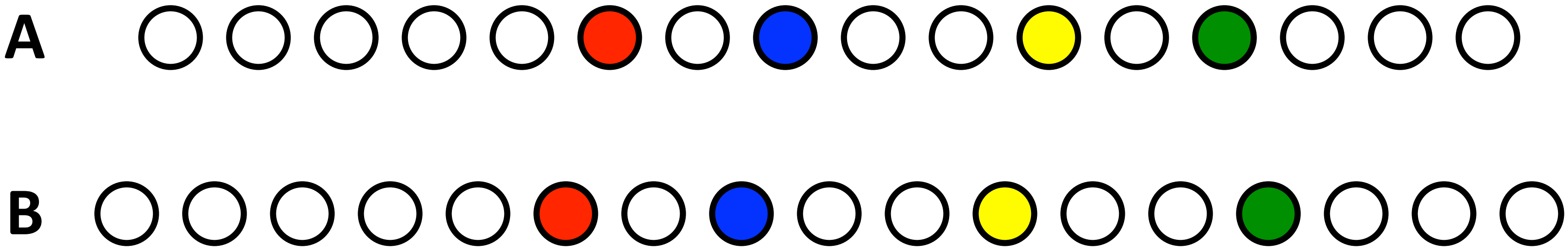


Round 4: ***Spoiler***

Winning strategy (by picture)



Round 4: ***Duplicator***

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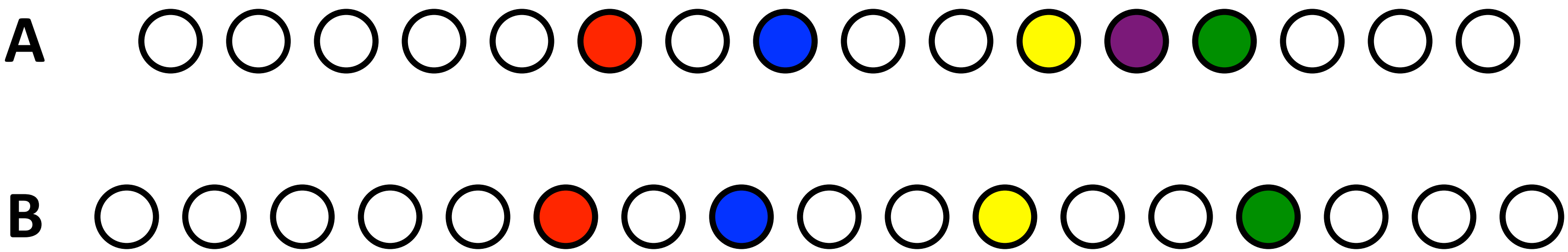
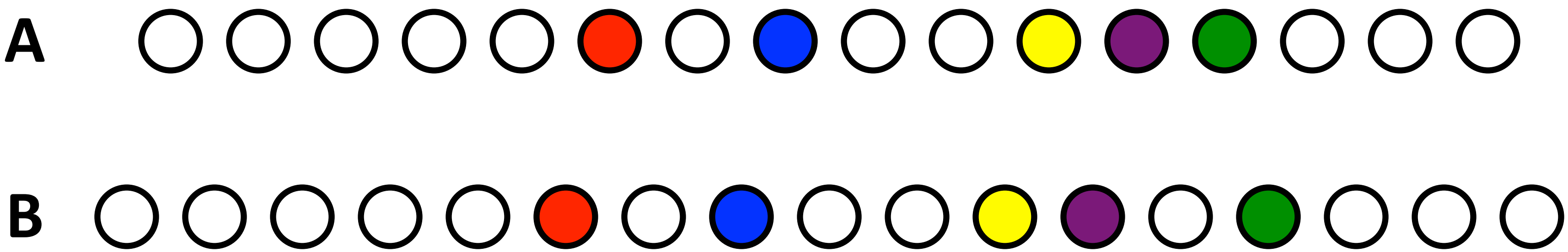


Round 5: ***Spoiler***

Winning strategy (by picture)



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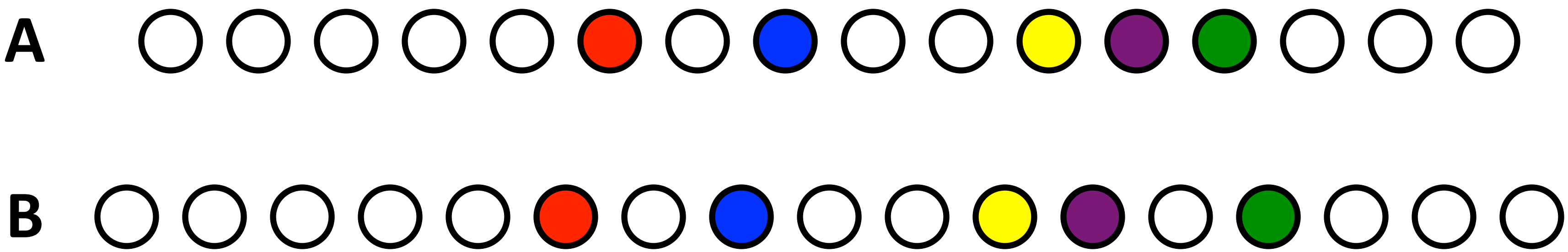

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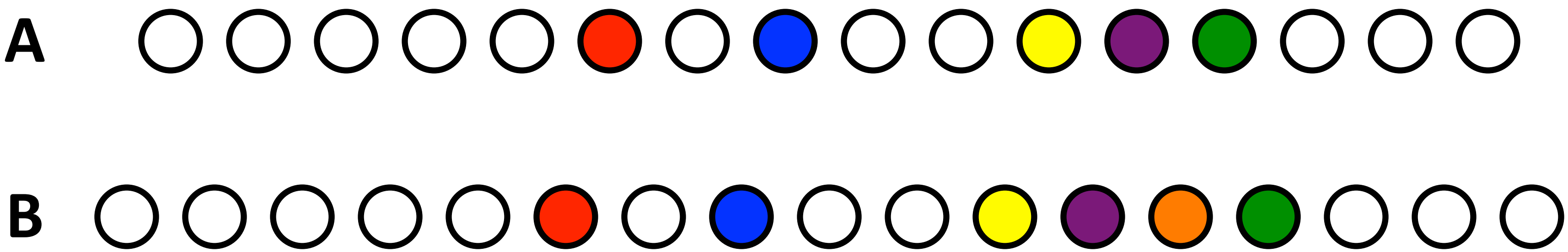

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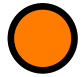
- However, ***Duplicator*** *loses* after ***Spoiler*** plays .

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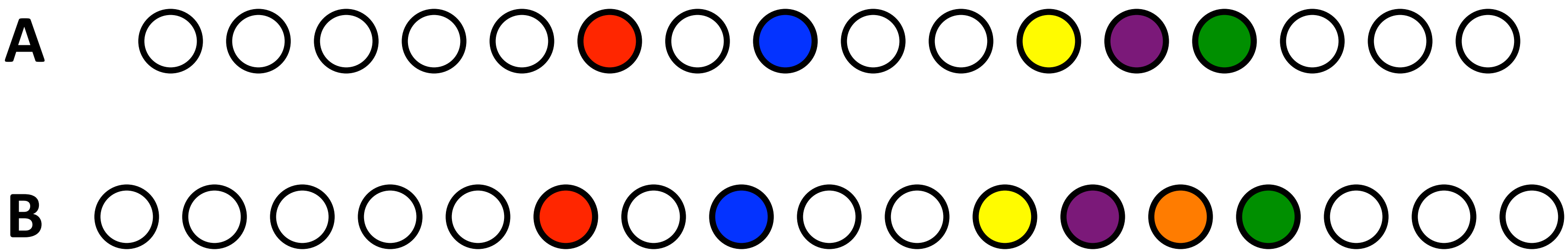

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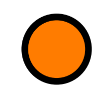
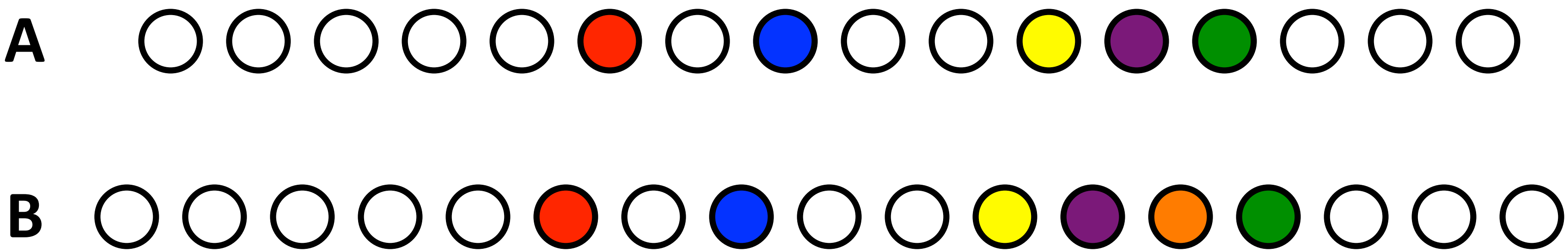
- However, ***Duplicator*** *loses* after ***Spoiler*** plays .

Winning strategy (by picture)



Duplicator's winning strategy:

in round j, preserve all distances between chosen elements up to 2^{k-j}

Connectivity

Corollary. GRAPH CONNECTIVITY is *not* FO definable

Connectivity

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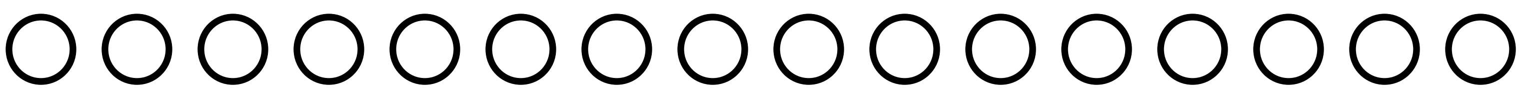
- If **A** is a linear order of size n, let G(**A**) be the graph with edges { i, i+2 mod n } for all a ∈ **A**



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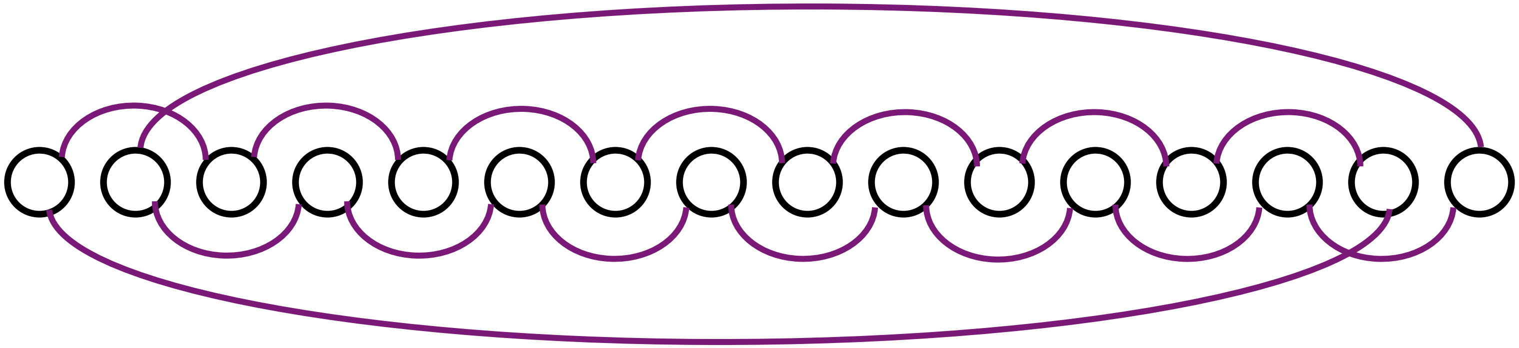

Obs 1: G(**A**) is connected if and only if n is odd

Corollary: … *not* FO definable

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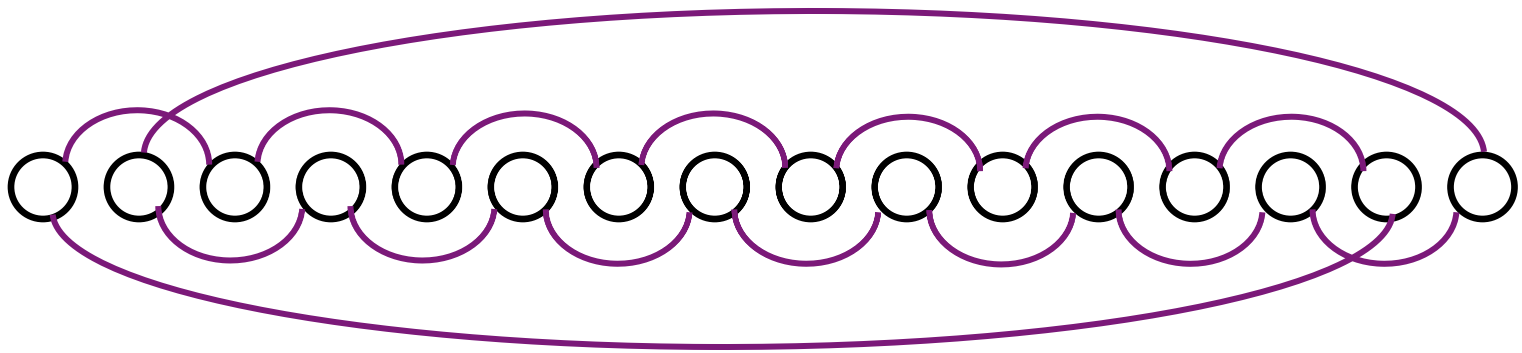

Obs 2: G(**A**) is first-order definable from **A**

Corollary: *not* FO definable

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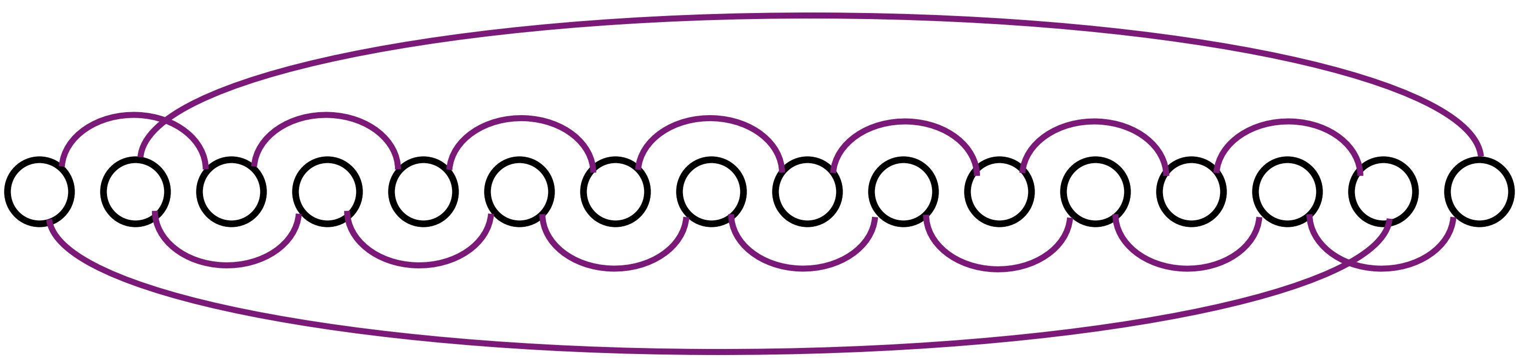


Connectivity

Corollary. GRAPH CONNECTIVITY is *not* FO definable

- If **A** is a linear order of size n, let G(**A**) be the graph with edges { i, i+2 mod n } for all a ∈ **A**

- If φ were a first-order formula defining GRAPH CONNECTIVITY, then by replacing each sub-formula E(x,y) with a formula "x and y have cyclic distance 2 in the linear order **A**", we could define EVENNESS of **A** (which we showed is impossible by the EF game).

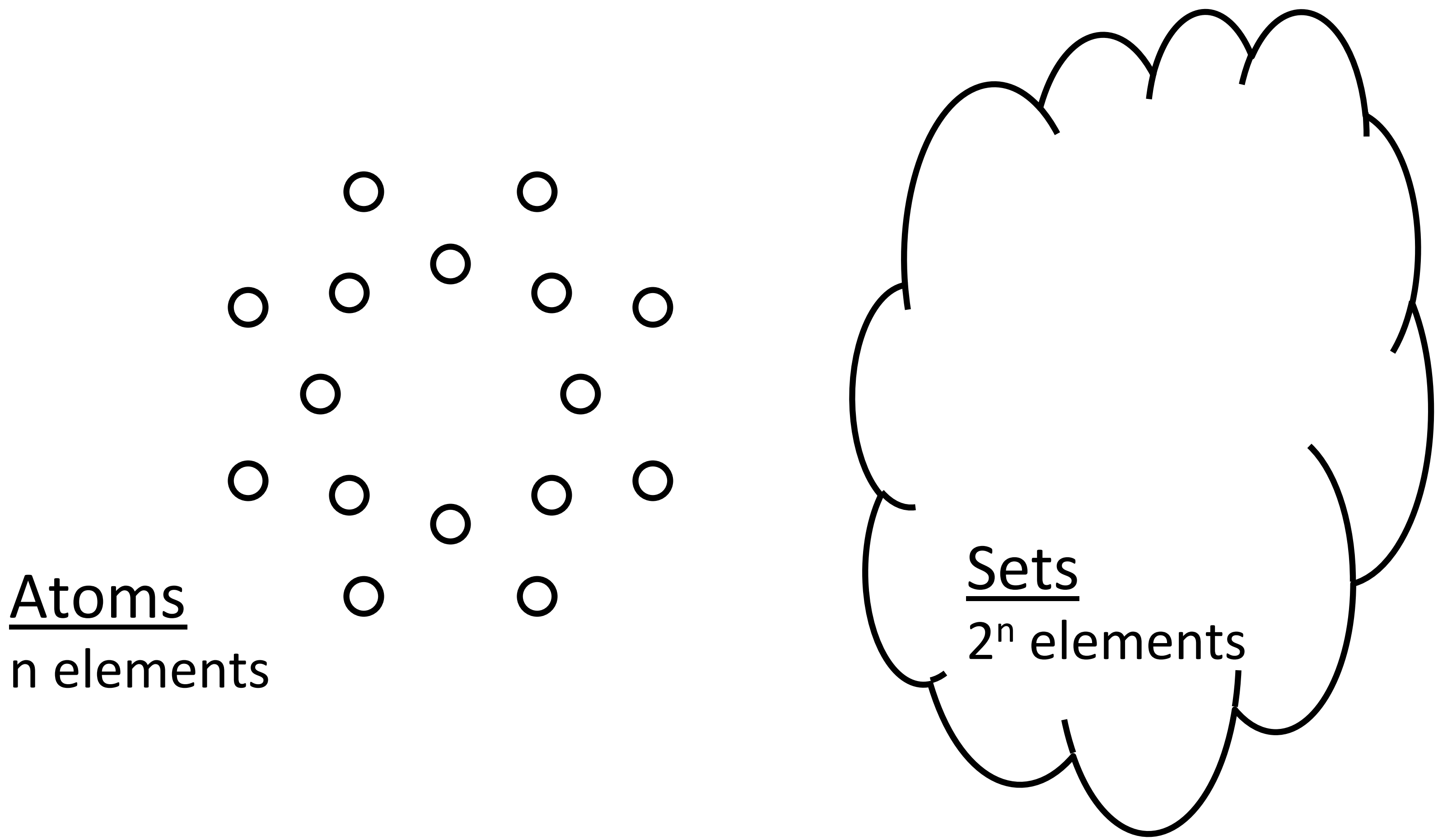
Connectivity

Corollary. GRAPH CONNECTIVITY is *not* FO definable

- This result can be proved directly by playing the EF game e.g. on graphs C_n and $C_n + C_n$
- The reduction to EVENNESS of linear orders illustrates the technique of a *first-order interpretations*.

Set-powersets

- SetPow_n is the structure $([n] \cup 2^{[n]}, \underline{\text{Atoms}}, \underline{\text{Sets}}, \underline{\text{In}})$



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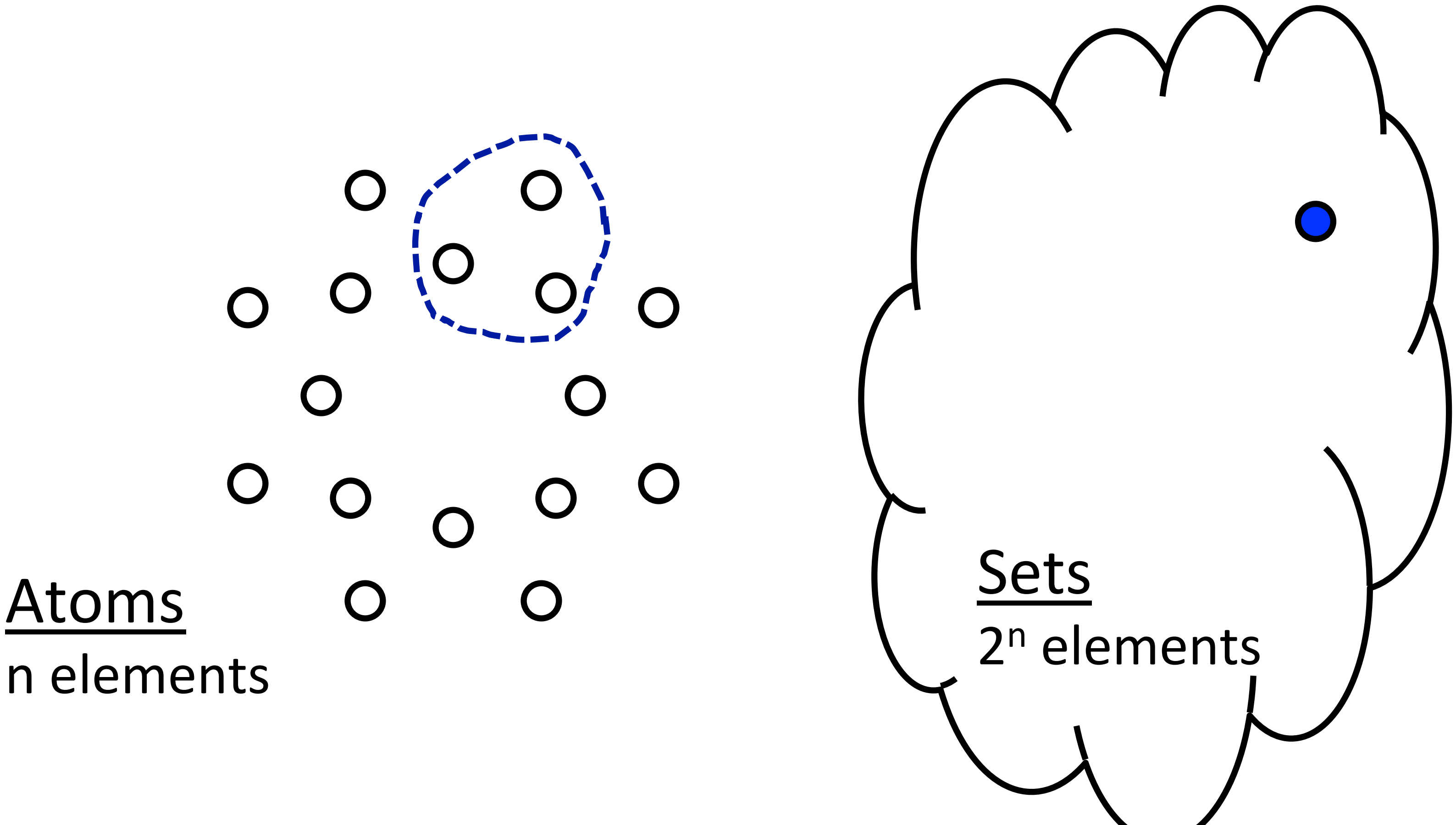

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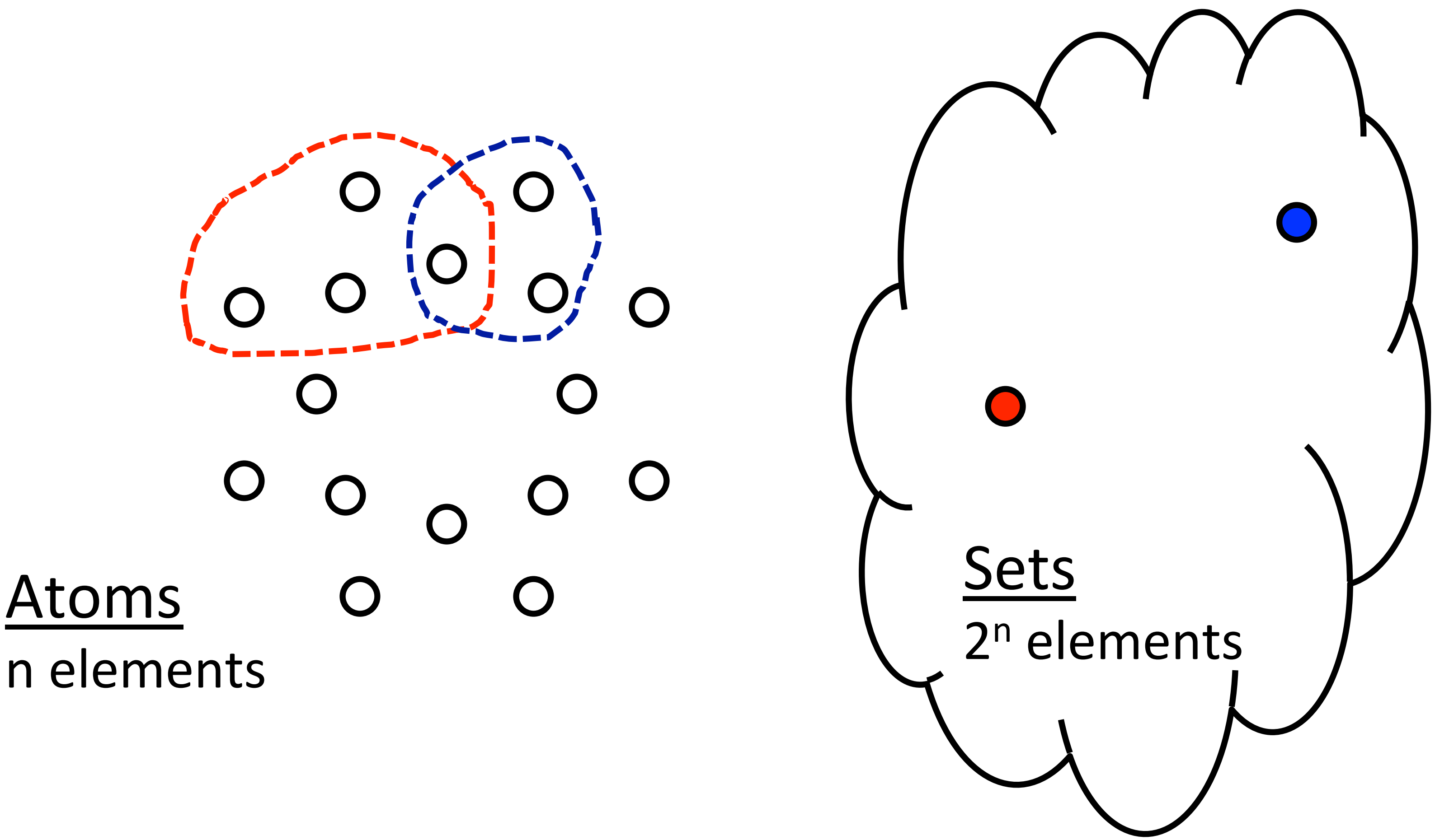

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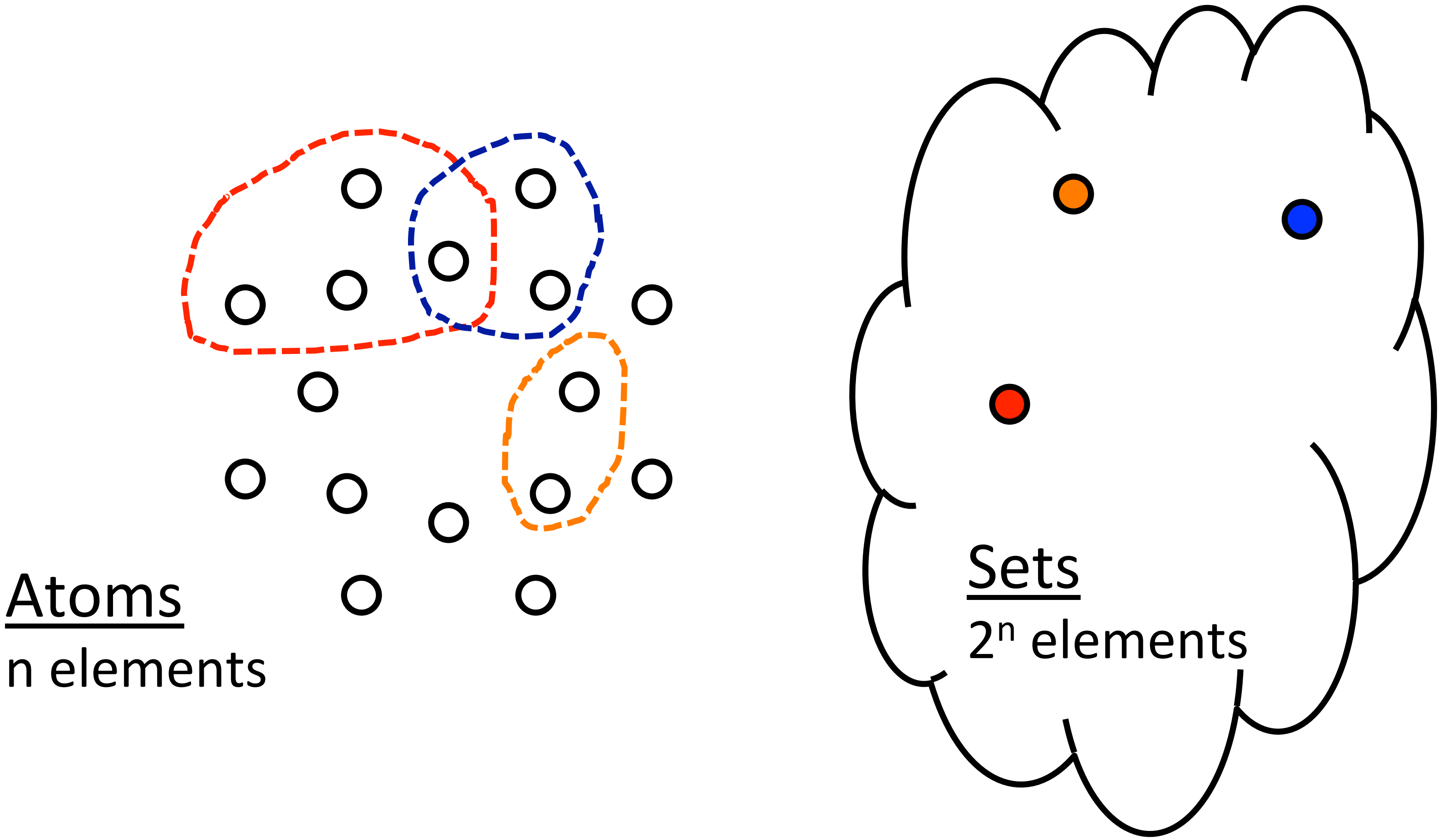

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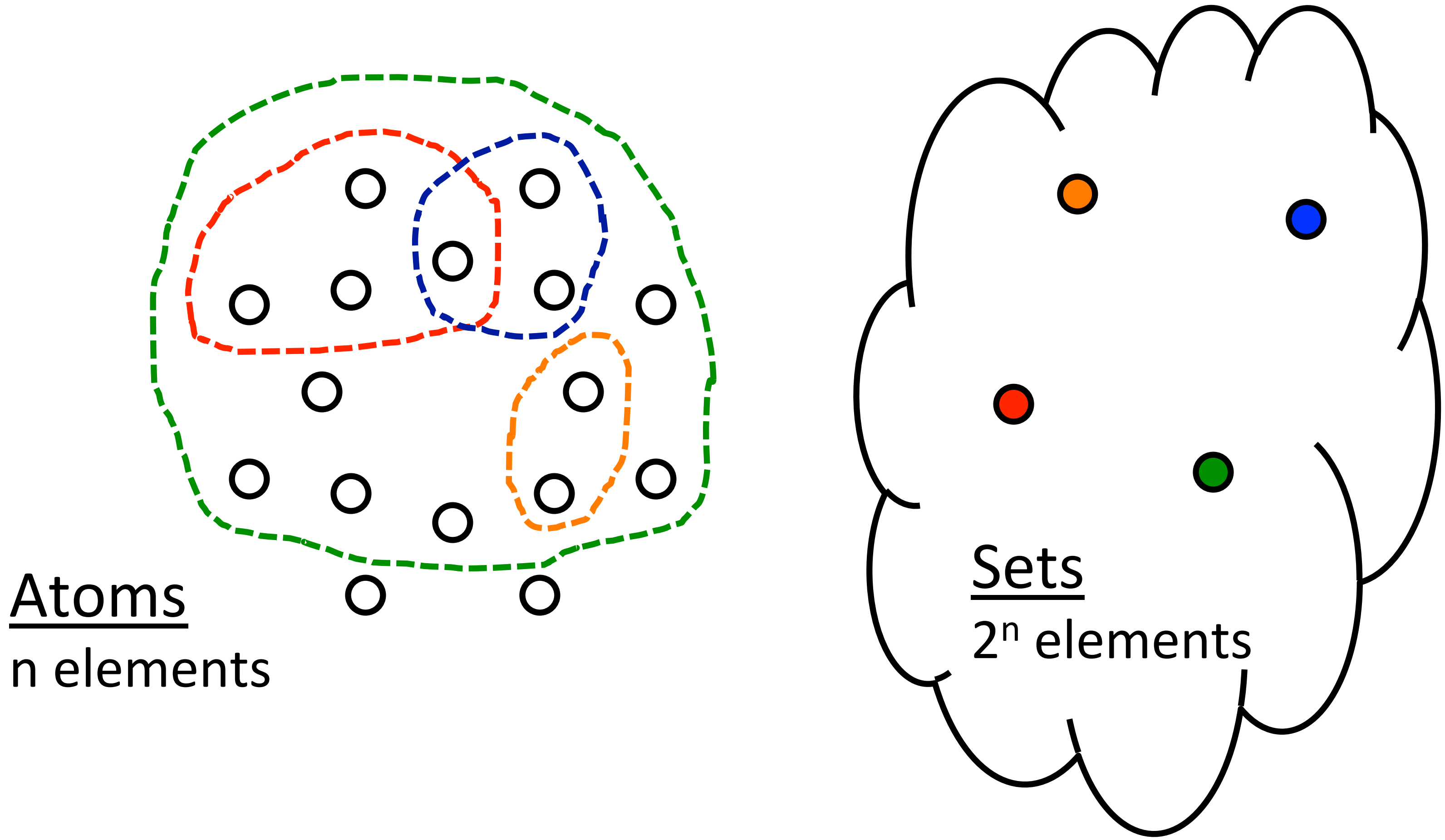

Set-powersets

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$$\underline{\text{Atoms}} = [n] = \{1,\ldots,n\},$$

$$\underline{\text{Sets}} = \text{powerset of } \underline{\text{Atoms}},$$

$$\underline{\text{In}} = \{(i,X) \in \underline{\text{Atoms}} \times \underline{\text{Sets}} \mid i \in X\}.$$

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- A **set-powerset** is any structure **A** with relations $\{\underline{\text{Atoms}}, \underline{\text{Sets}}, \underline{\text{In}}\}$ which is isomorphic to SetPow_n for some $n > 0$. It is said to be EVEN/ODD according to the parity of n.

Set-powersets

Obs. The class of set-powersets is FO definable.

- We *cannot* say (in first-order logic):

$\forall X \subseteq \underline{\text{Atoms}}\ \exists S \in \underline{\text{Sets}}\ \forall x \in \underline{\text{Atoms}},\ x \in X \Leftrightarrow \underline{\text{In}}(x,S)$

- Instead, we say:

"$\emptyset \in \underline{\text{Sets}}$" $\wedge$ $\forall S \in \underline{\text{Sets}}\ \forall x \in \underline{\text{Atoms}}$ "$S \cup \{x\} \in \underline{\text{Sets}}$"

- This formula exploits *finiteness* in an essential way.

Set-powersets

Theorem

The class of EVEN set-powersets is not FO definable.

Set-powersets

Theorem

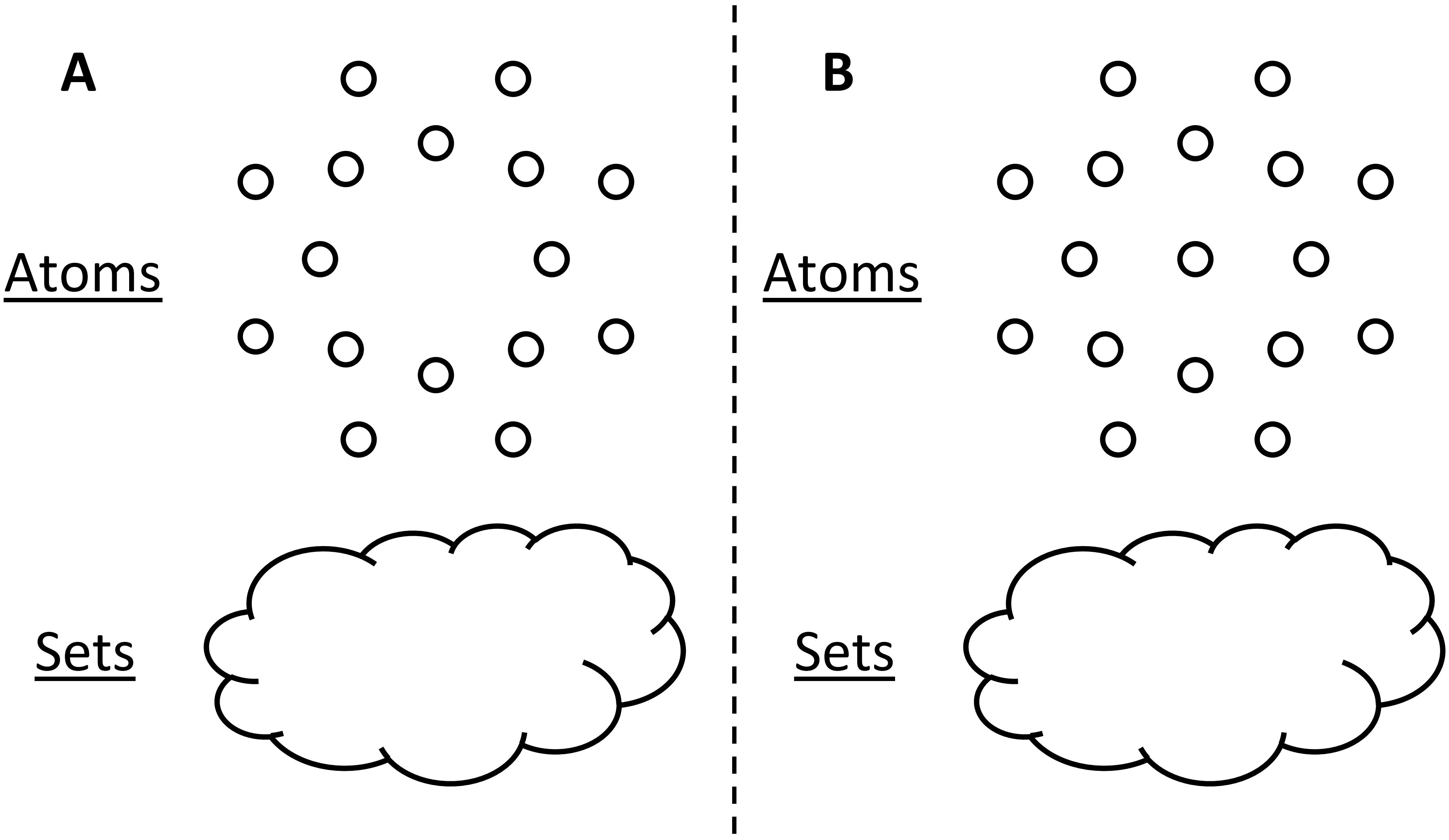
The class of EVEN set-powersets is not FO definable.

Proof

For every k, we show that ***Duplicator*** has a winning strategy in the k-round Ehrenfeucht-Fraisse game on

$$\mathbf{A} = \text{SetPow}_{2\text{^}k} \quad \text{and} \quad \mathbf{B} = \text{SetPow}_{2\text{^}k+1}$$

Winning strategy (by picture)



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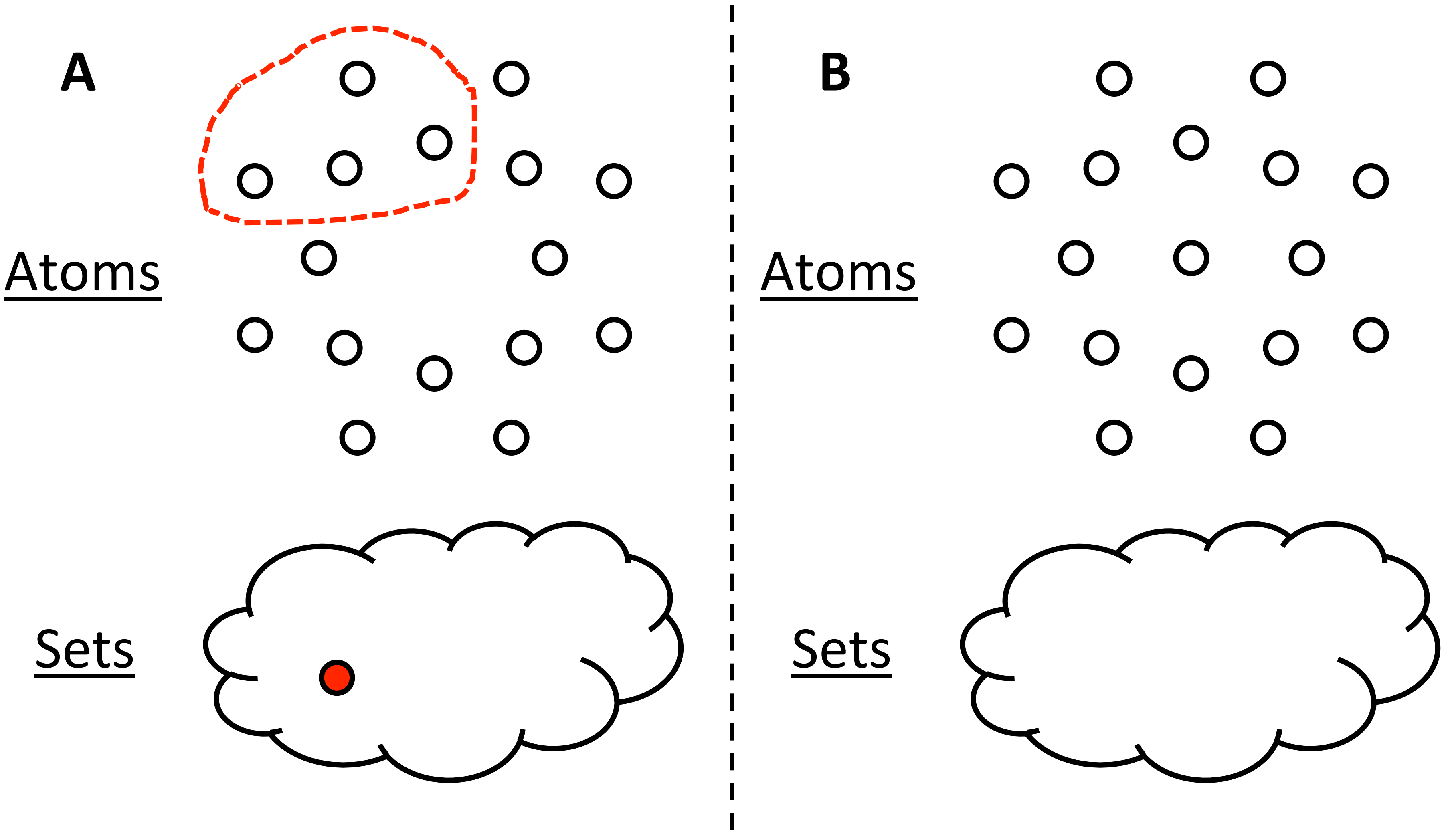

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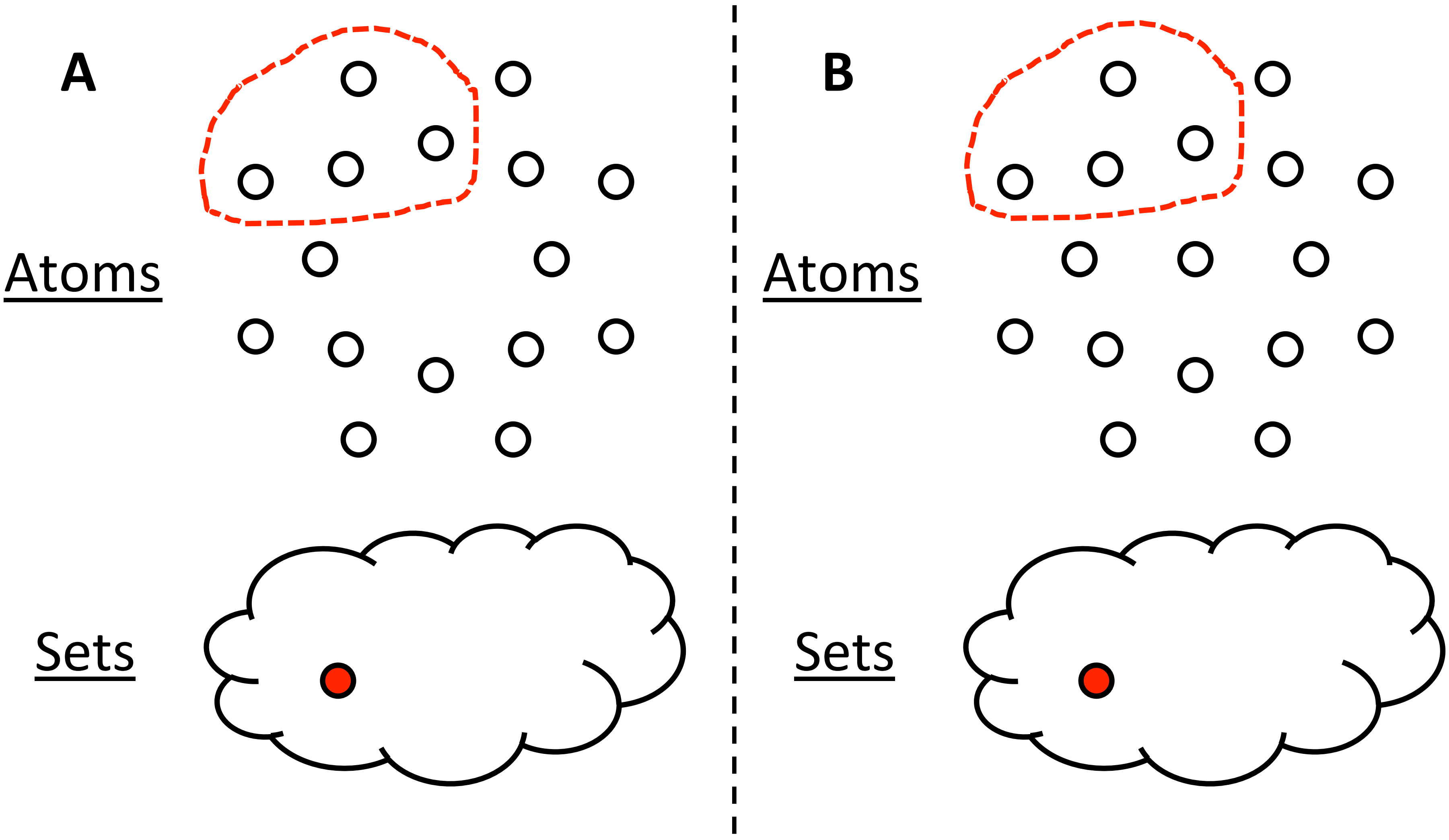

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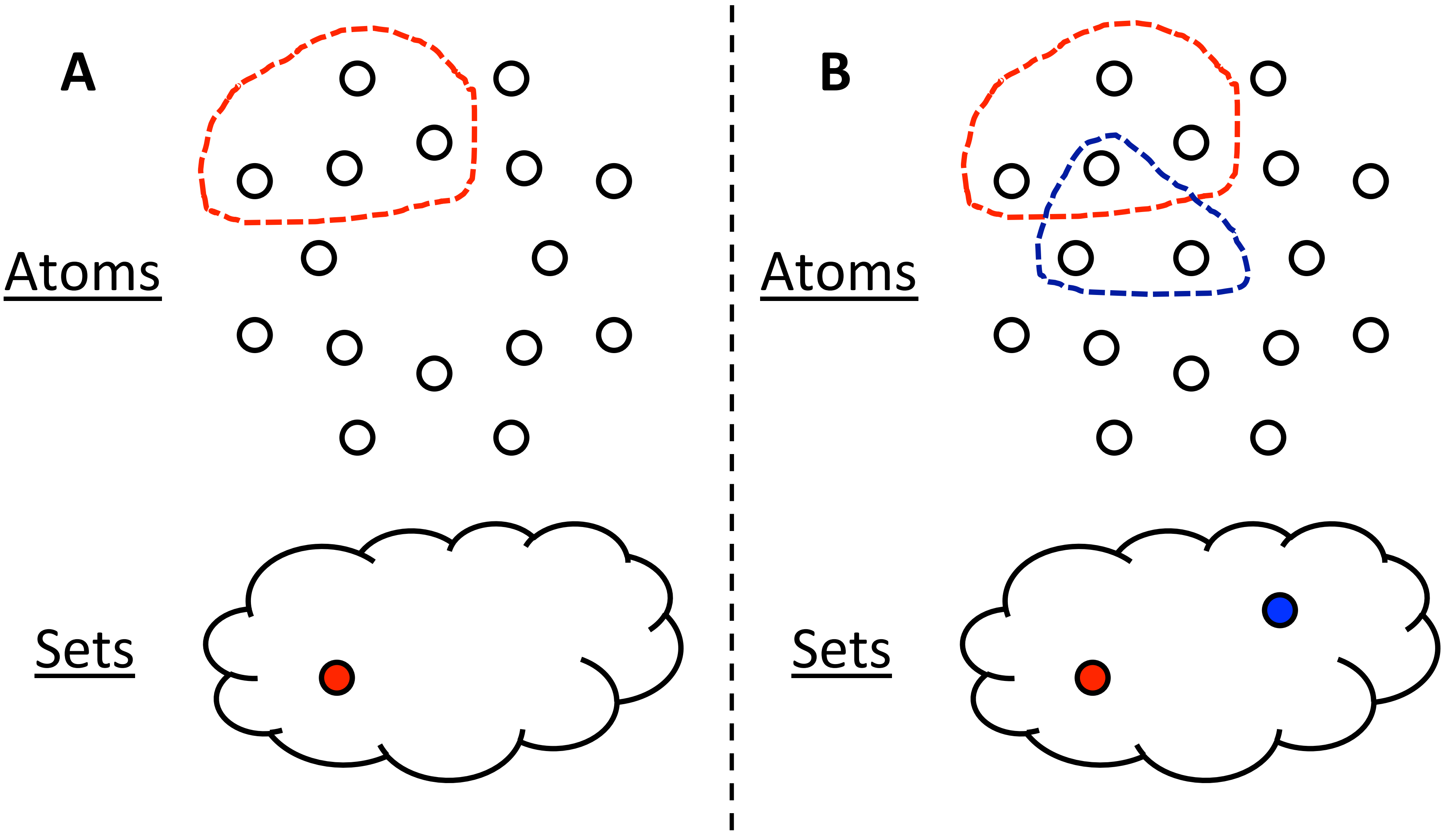

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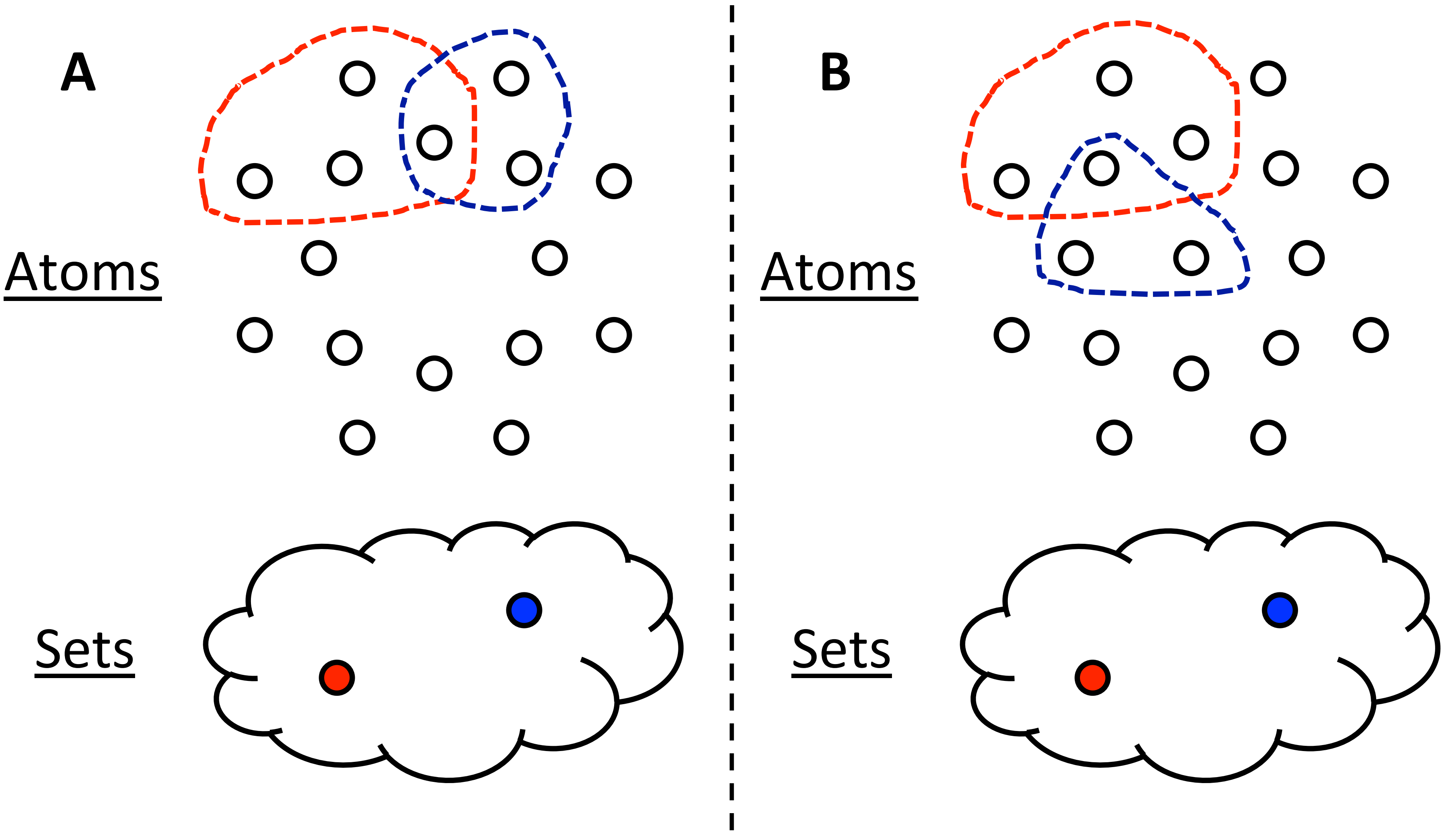

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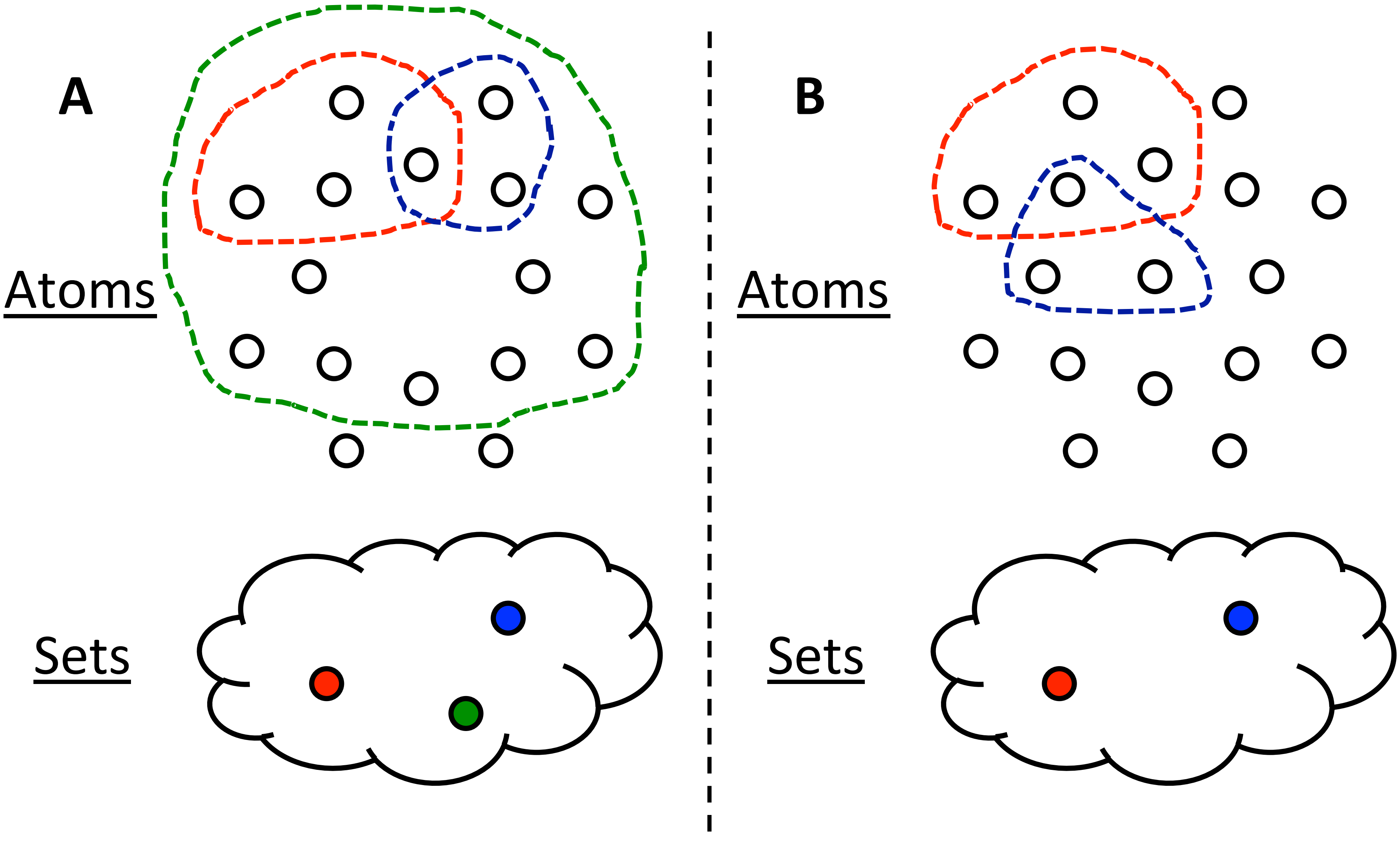

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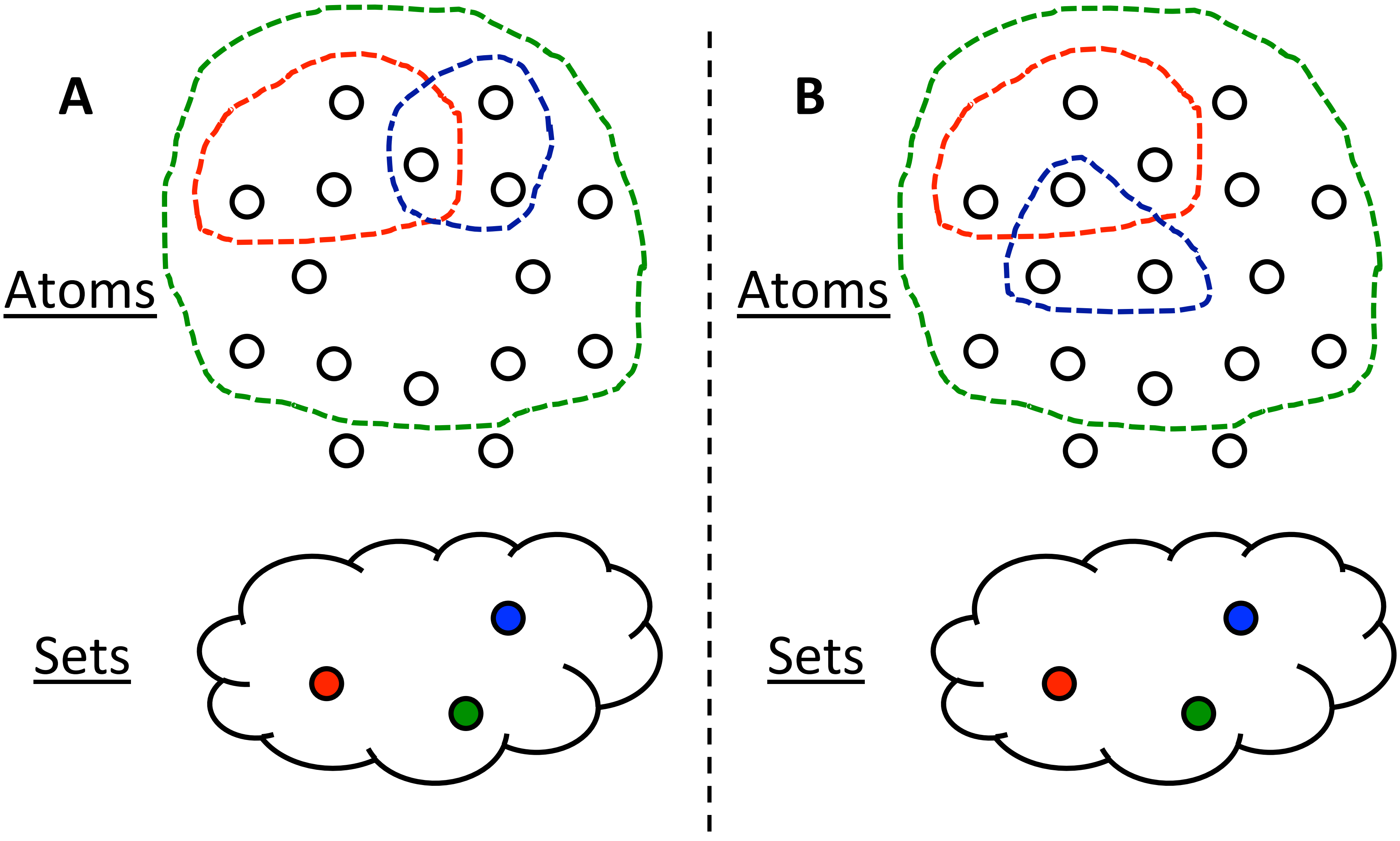

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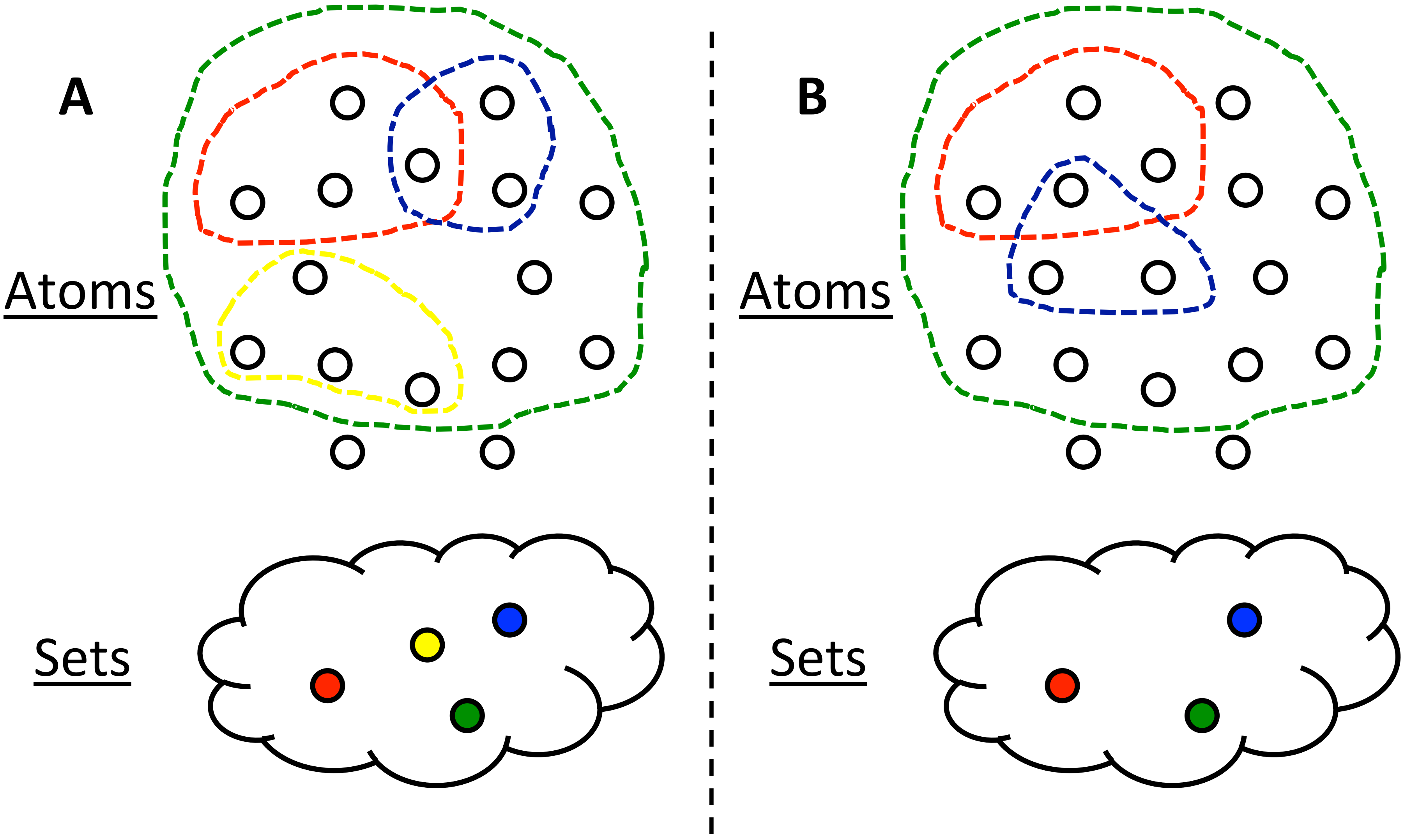

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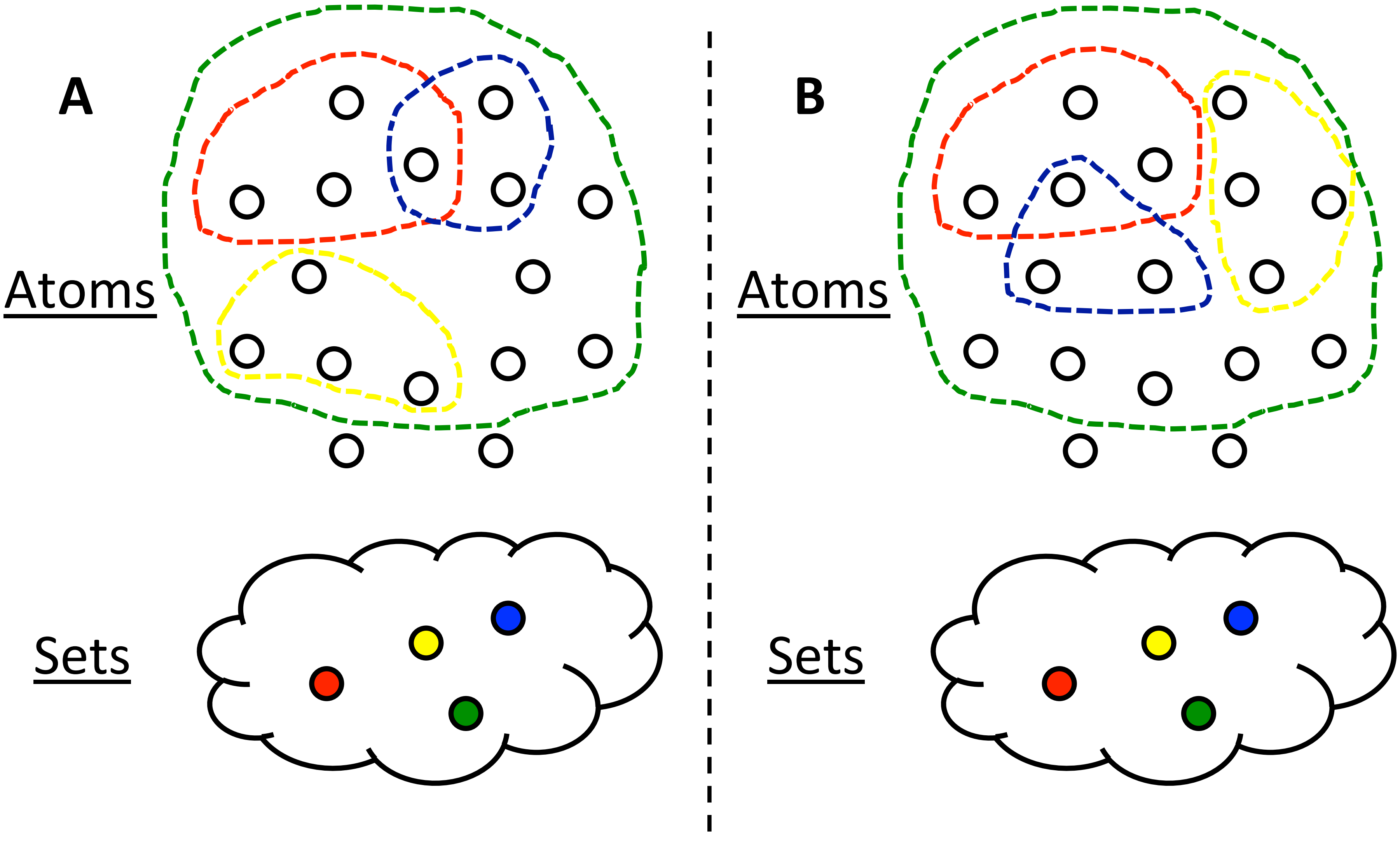

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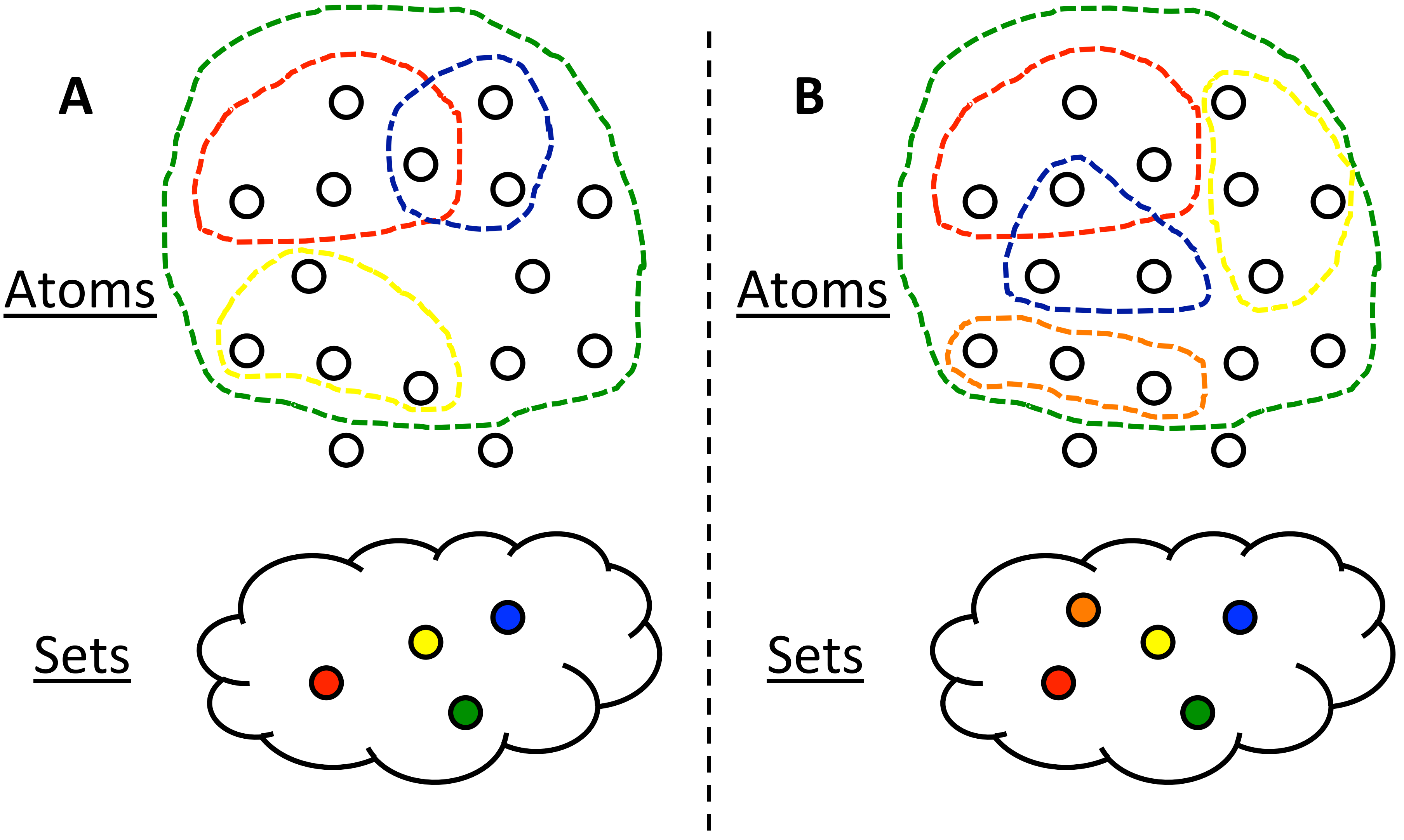

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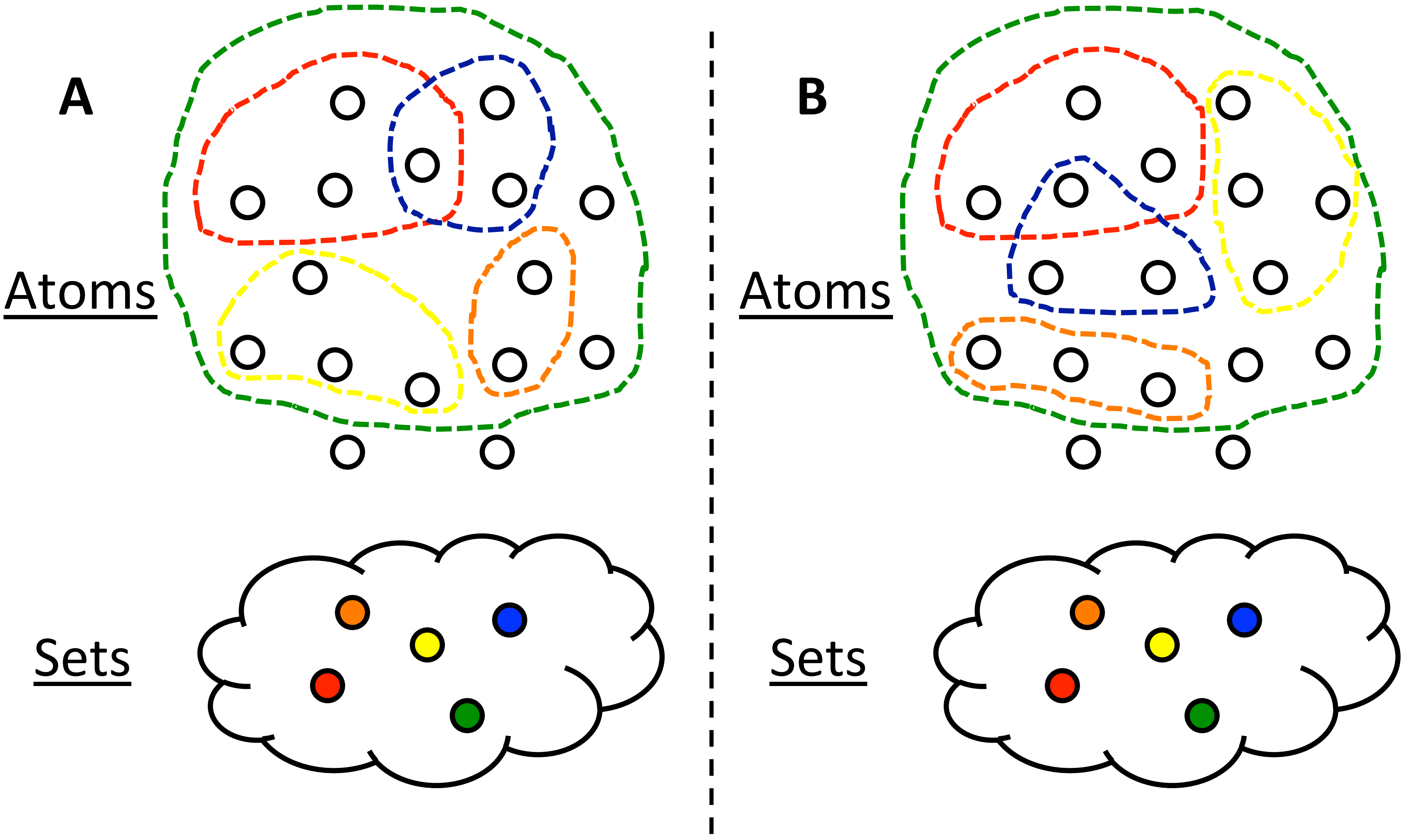

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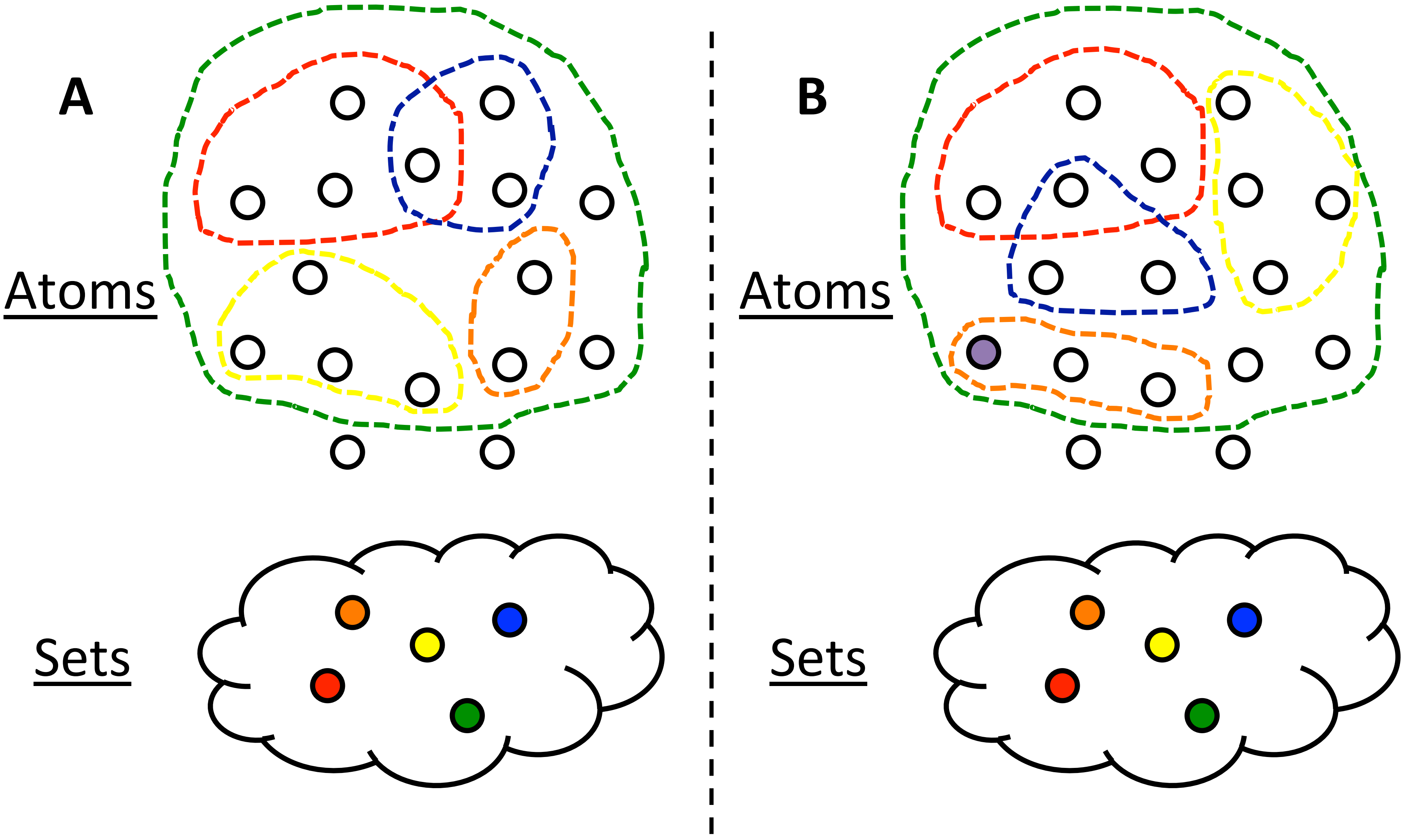

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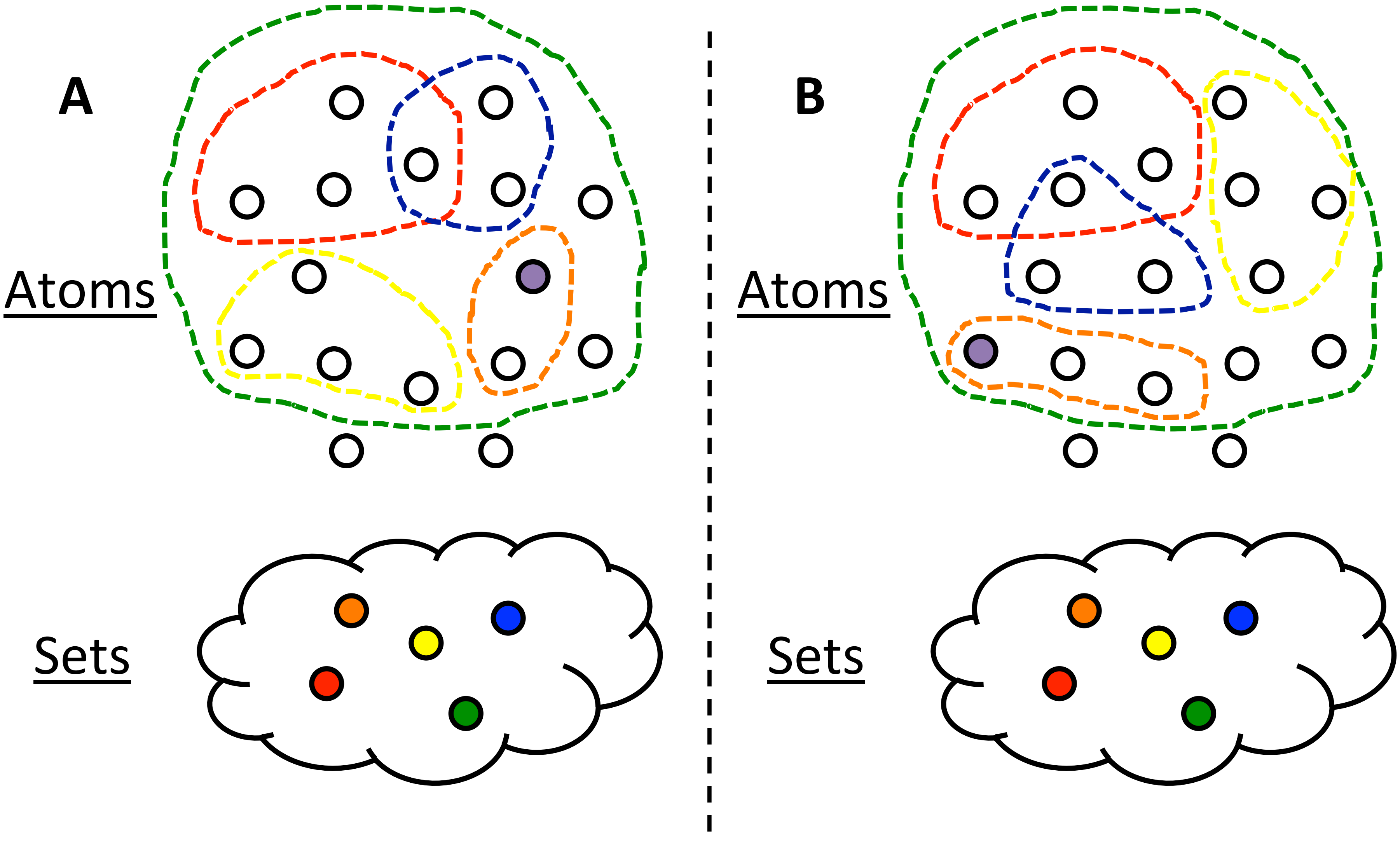

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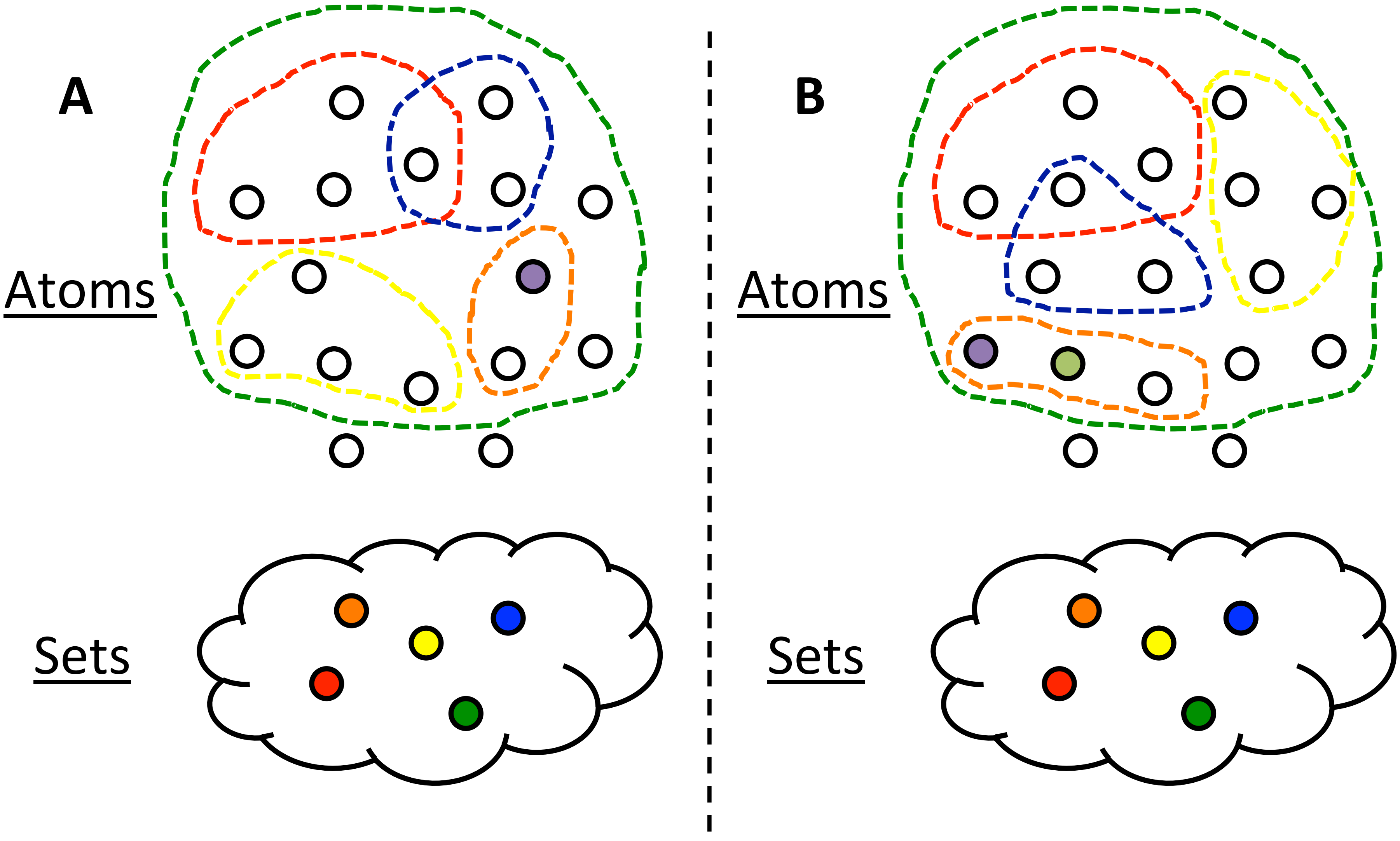

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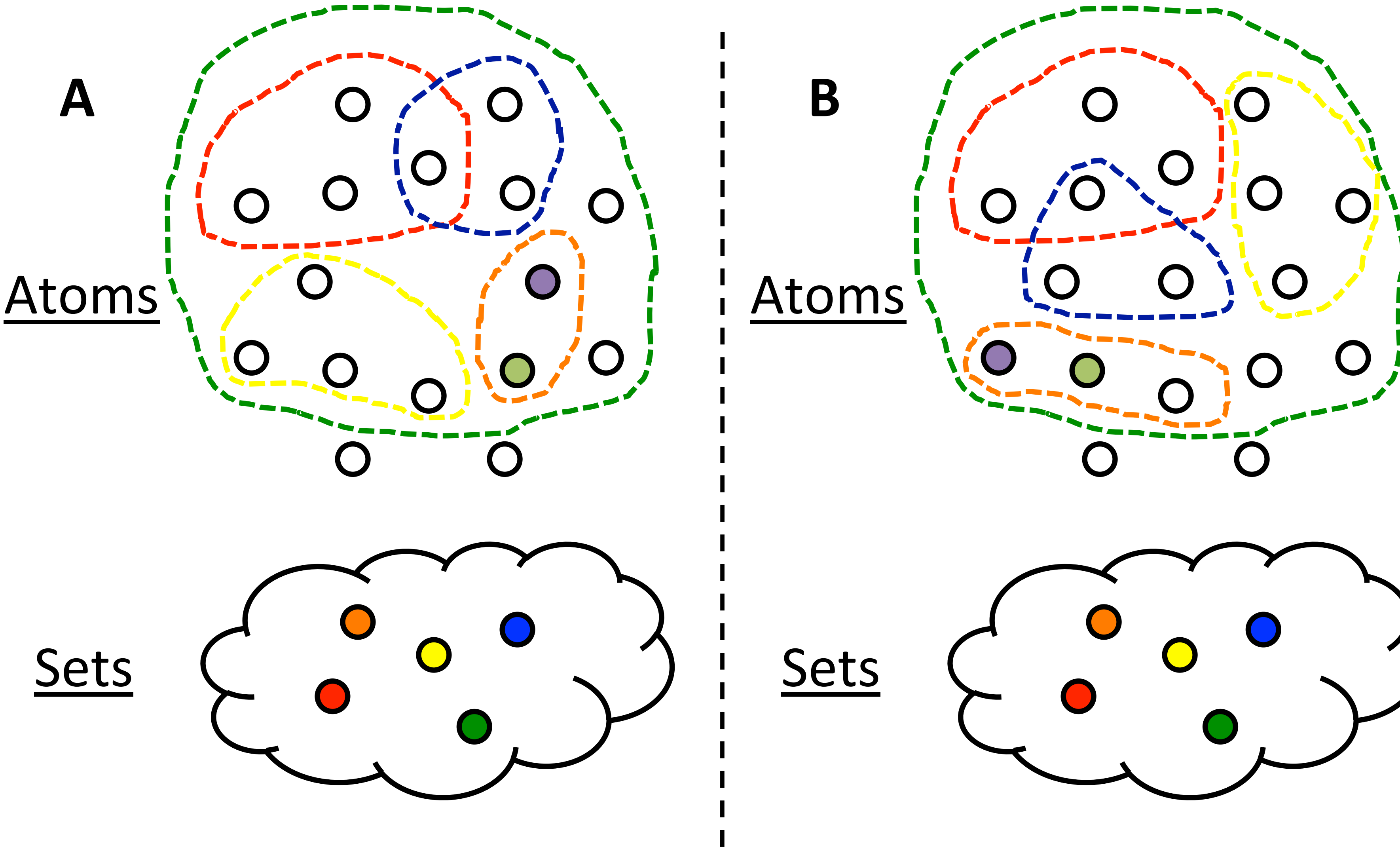

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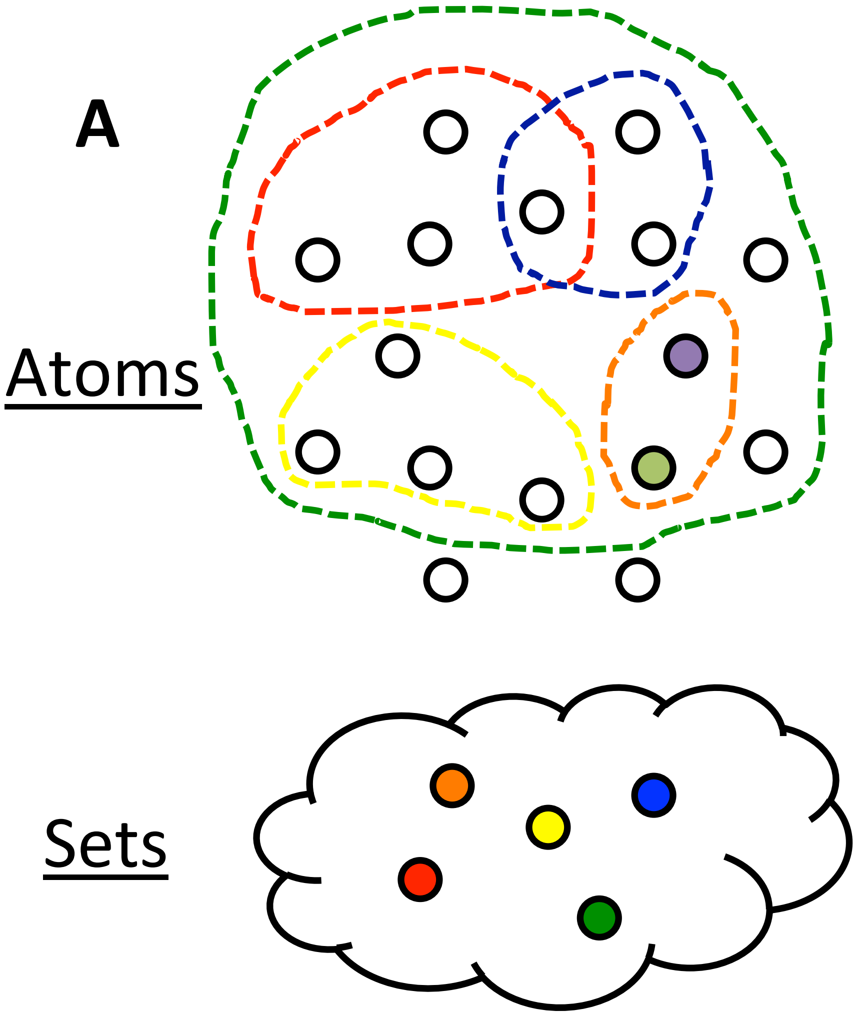

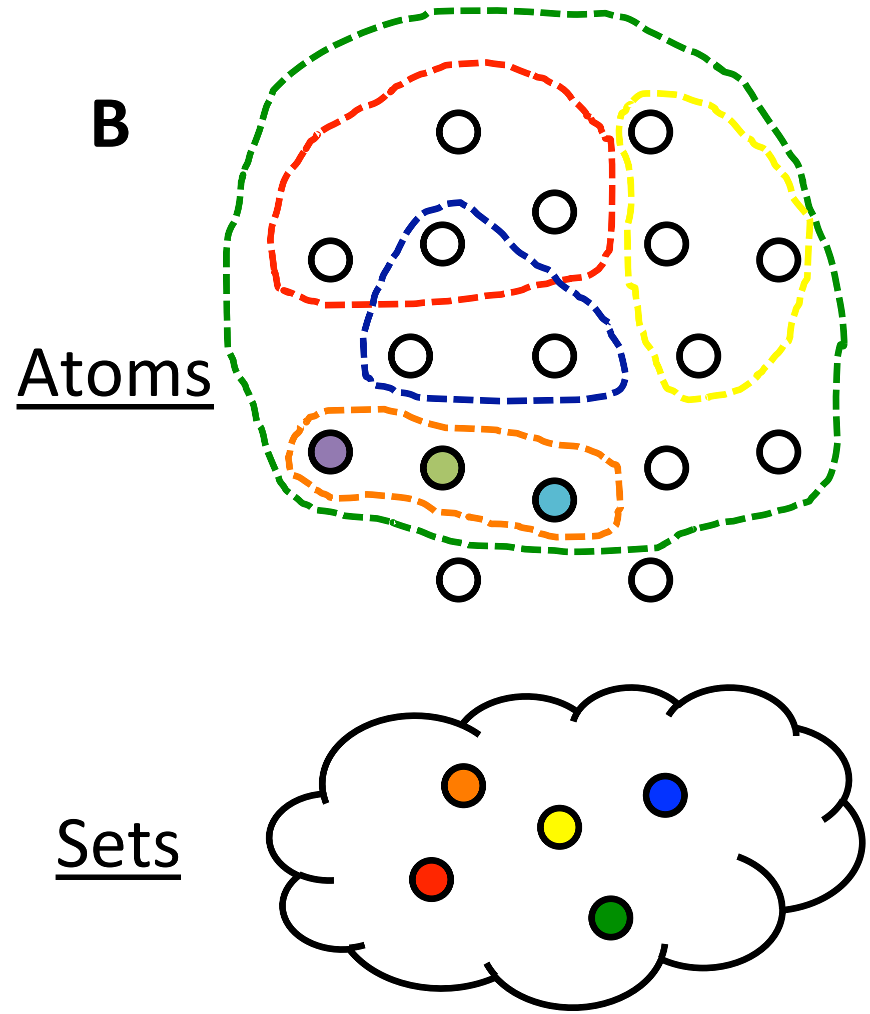

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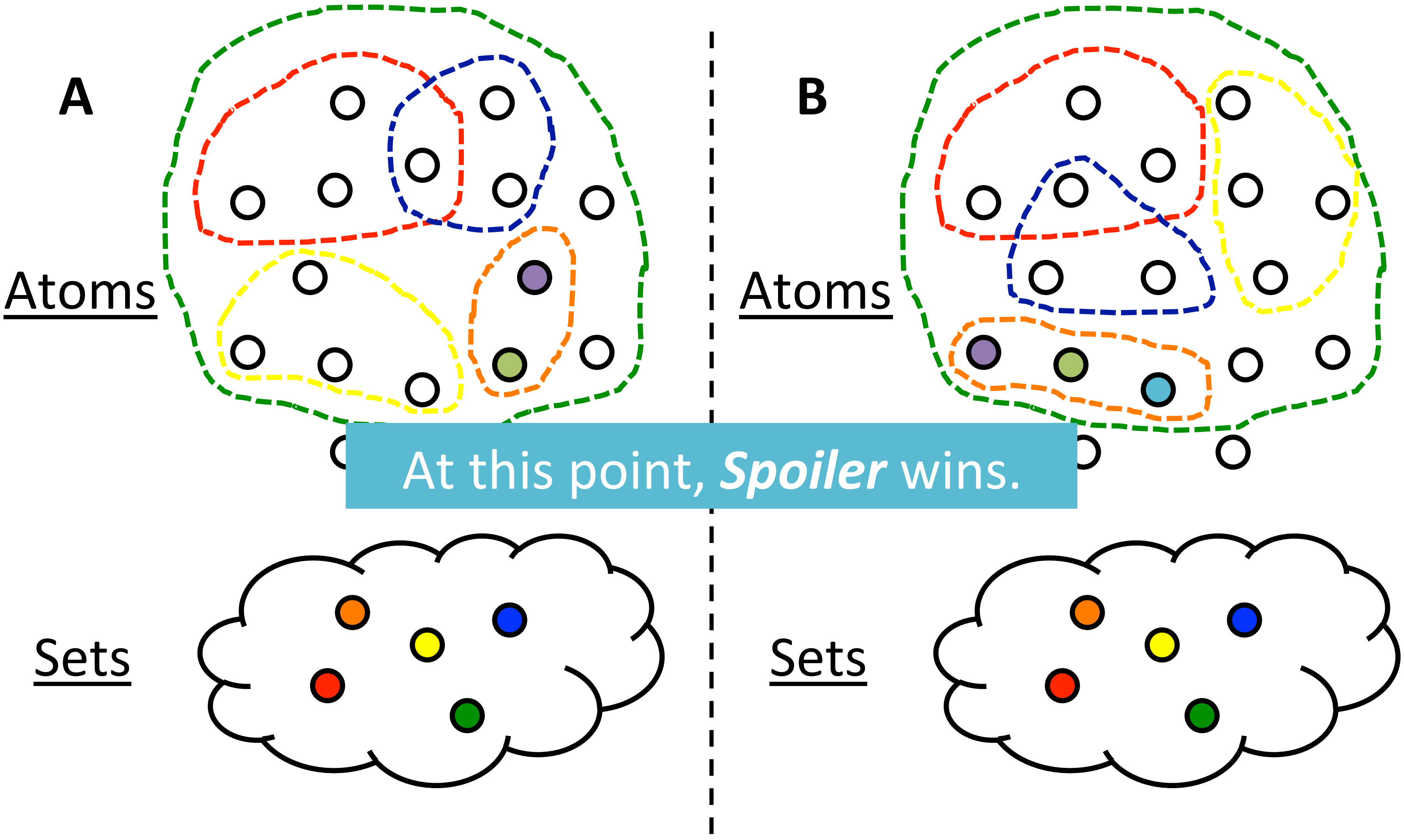

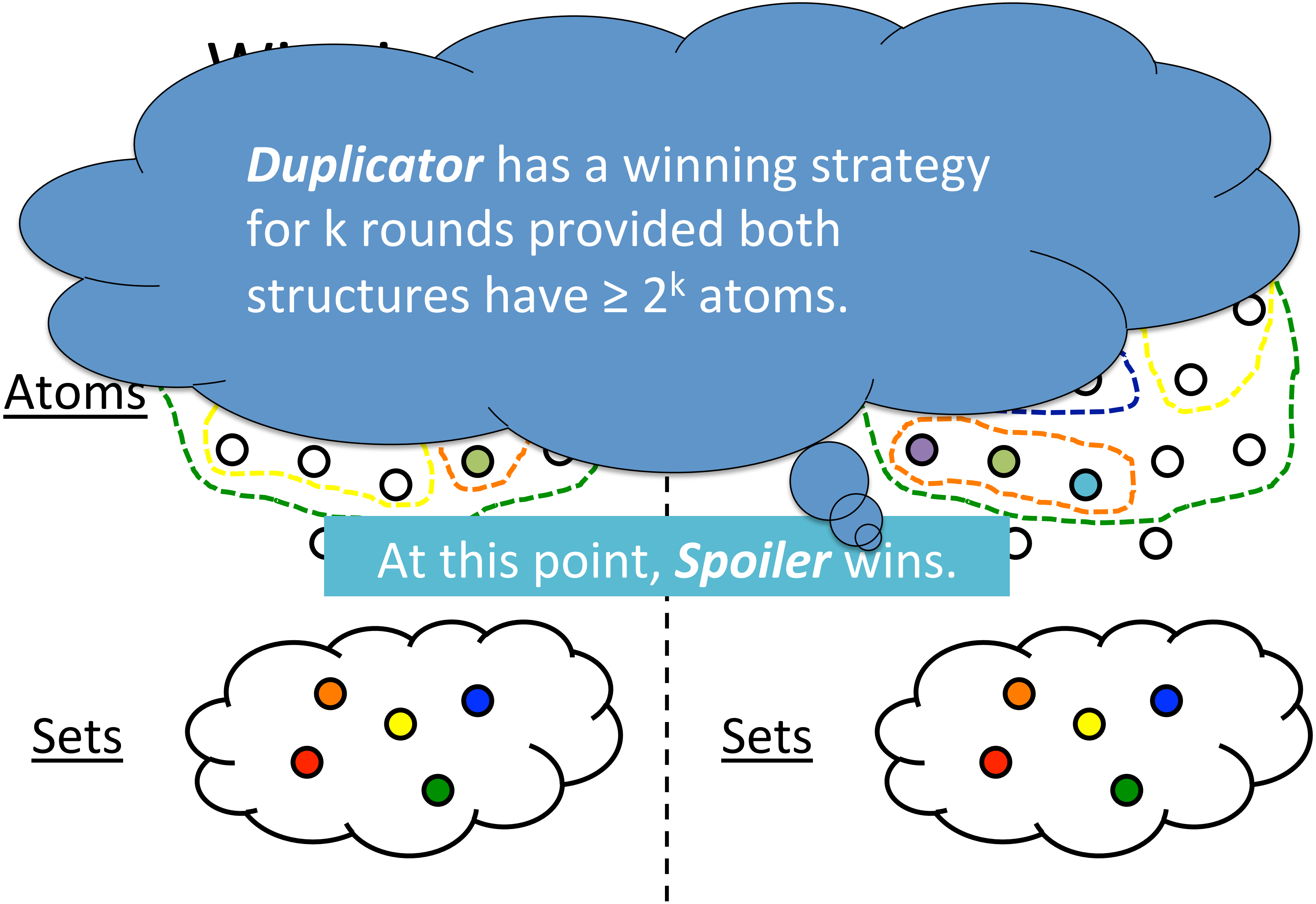
Duplicator has a winning strategy for k rounds provided both structures have ≥ 2^k atoms.
Atoms
At this point, *Spoiler* wins.
Sets
Sets

Duplicator has a winning strategy for k rounds provided both structures have $\geq 2^k$ atoms.

Atoms

***Duplicator's* winning strategy**:

in round j, preserve the cardinality up to 2^{k-j} of every Boolean combination of the chosen sets and atoms