Next Steps (Section 5.8)

Δ-definability of sets

$$\textsc{Terms} := \{\ulcorner t \urcorner : \text{terms } t\} = \{a \in \mathbb{N} : a = \ulcorner t \urcorner \text{ for some term } t\},$$

$$\textsc{Formulas} := \{\ulcorner \varphi \urcorner : \text{formulas } \varphi\} = \{a \in \mathbb{N} : a = \ulcorner \varphi \urcorner \text{ for some formula } \varphi\}.$$

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$\ulcorner \neg\alpha \urcorner = \langle 1, \ulcorner \alpha \urcorner \rangle$	$\ulcorner {=} t_1 t_2 \urcorner = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner + t_1 t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner < t_1 t_2 \urcorner = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
$\ulcorner (\alpha \vee \beta) \urcorner = \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle$	$\ulcorner 0 \urcorner = \langle 9 \rangle$	$\ulcorner \cdot\, t_1 t_2 \urcorner = \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner v_i \urcorner = \langle 2i \rangle$
$\ulcorner (\forall v_i)(\alpha) \urcorner = \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle$	$\ulcorner St \urcorner = \langle 11, \ulcorner t \urcorner \rangle$	$\ulcorner E t_1 t_2 \urcorner = \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	

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Recall the inductive definition of an $\mathcal{L}_{NT}$-**term** t: it is either

- a variable symbol v_i,
- the constant symbol 0,
- St_1 where t_1 is term,
- $+t_1t_2$ or $\cdot\, t_1t_2$ or Et_1t_2 where t_1, t_2 are terms.

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Let's start with Δ-definition of

$$\text{Variables} := \{\ulcorner v_i \urcorner : i = 1, 2, \ldots\} \quad (= \{2^{2i+1} : i = 1, 2, \ldots\}).$$

by the formula

$$\mathit{Variable}(x) :\equiv (\exists y < x)[\mathit{Even}(y) \wedge (0 < y) \wedge (x = 2^{Sy})].$$

Δ-Definition of $\text{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

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Recall the inductive definition of an $\mathcal{L}_{NT}$**-term** t: it is either

- a variable symbol v_i,
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- $+t_1t_2$ or $\cdot\, t_1t_2$ or Et_1t_2 where t_1, t_2 are terms.

We would like to write:

$$Term(x) :\equiv Variable(x) \vee \overbrace{x = \overline{2^{10}}}^{\text{“}x \text{ is } \ulcorner 0 \urcorner\text{”}} \vee \overbrace{(\exists y < x)[Term(y) \wedge x = \underbrace{\overline{2^{12} \cdot 3^{Sy}}}_{\langle 11, y \rangle}]}^{\text{“}x \text{ is } \ulcorner St_1 \urcorner \text{ for some term } t_1\text{”}}$$
$$\vee \underbrace{\cdots}_{\text{“}x \text{ is } +t_1t_2 \text{ or } \cdot\, t_1t_2 \text{ or } Et_1t_2\text{”}}$$

However, there is a problem with this “Δ-formula”.

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$$\vee \quad \underbrace{\cdots}_{\text{“}x \text{ is } +t_1t_2 \text{ or } \cdot\, t_1t_2 \text{ or } Et_1t_2\text{”}}$$

This is a not legitimate formula of first-order logic! Note the circular use of the subformula $Term(y)$.

Δ-Definition of $\text{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

Definition. A *term construction sequence* for a term t is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell :\equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term t_k is either

- a variable symbol,
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Example. $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

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Example. $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

Lemma. Every term t has a term construction sequence of length at most the number of symbols in t.

(Easy proof by induction.)

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Key to defining $\textsc{Terms}$**:** We will write a Δ-formula defining the set

$$\textsc{TermConSeq} = \{(c, a) : c = \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle \text{ and } a = \ulcorner t_\ell \urcorner \text{ where } (t_1, \ldots, t_\ell) \text{ is a term construction sequence}\}.$$

Δ-Definition of $\textsc{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

$$
\begin{aligned}
&TermConSeq(c,a) :\equiv \\
&\quad Codenumber(c) \wedge (\exists \ell < c)\Big[Length(c,\ell) \wedge IthElement(a,\ell,c) \wedge \\
&\qquad (\forall k \leq \ell)(\exists e_k < c)\Big[IthElement(e_k,k,c) \wedge \\
&\qquad\quad \begin{pmatrix} Variable(e_k) \\ \vee\ e_k = \overline{2^{10}} \ \Big\}\ \text{“}e_k \text{ is } \ulcorner 0 \urcorner\text{”} \\ \vee\ (\exists j < k)(\exists e_j < c)[IthElement(e_j,j,c) \wedge \overbrace{e_k = \overline{2^{12}} \cdot \overline{3}^{Se_j}}^{\text{“}e_k \text{ is } \ulcorner Se_j \urcorner\text{”}}] \\ \vee \cdots \end{pmatrix} \Big]\Big]
\end{aligned}
$$

Key to defining $\textsc{Terms}$**:** We will write a Δ-formula defining the set

$$
\begin{aligned}
\textsc{TermConSeq} = \{(c,a) : c = \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle \text{ and } a = \ulcorner t_\ell \urcorner \text{ where} \\
(t_1, \ldots, t_\ell) \text{ is a term construction sequence}\}.
\end{aligned}
$$

Δ-Definition of $\text{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

Now there is an obvious way to define $Term(a)$:

$$Term(a) :\equiv (\exists c)\, TermConSeq(c, a).$$

To make this a Δ-formula, we need an upper bound on c as a function of a.

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Now there is an obvious way to define $\mathit{Term}(a)$:

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To make this a Δ-formula, we need an upper bound on c as a function of a.

Suppose $a = \ulcorner t \urcorner$. Another easy lemma by induction: The number of symbols in t is at most a. Therefore, there exists a term construction sequence $(t_1, \ldots, t_\ell)$ for t with length $\leq a$. We may assume that each t_k is a subterm of t, so that $\ulcorner t_k \urcorner \leq \ulcorner t \urcorner = a$ for all $k \in \{1, \ldots, \ell\}$.

Let $c := \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle$. We have

$$c = 2^{\ulcorner t_1 \urcorner + 1} 3^{\ulcorner t_2 \urcorner + 1} \cdots (p_\ell)^{\ulcorner t_\ell \urcorner + 1} \leq (p_\ell)^{\ulcorner t_1 \urcorner + \cdots + \ulcorner t_\ell \urcorner + \ell} \leq (p_\ell)^{\ell a + \ell} \leq (p_a)^{a^2 + a}.$$

Easy fact: The a^{th} prime number p_a is at most a^a. (In fact, $p_a \leq 2^{a^2}$ using the Prime Number Theorem: $a(\log a + \log\log a - 1) < p_a < a(\log a + \log\log a)$ for all $a \geq 6$.) We conclude that $c \leq a^{a(a^2+a)} \leq a^{2a^3}$.

Δ-DEFINITION OF TERMS $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

Now there is an obvious way to define $Term(a)$:

$$Term(a) :\equiv (\exists c)\, TermConSeq(c, a).$$

To make this a Δ-formula, we need an upper bound on c as a function of a.

We may therefore take

$$Term(a) :\equiv (\exists c \leq \underbrace{Ea{\cdot}SS0EaSSS0}_{a^{2a^3}})\, TermConSeq(c, a).$$

Construction Sequences for Other Recursive Definitions

In a similar way, using the notion of a *formula construction sequence*, we get a Δ-definition of the set

$$\textsc{Formulas} = \{\ulcorner\varphi\urcorner : \varphi \text{ is a formula}\}.$$

Definition. A *formula construction sequence* for a formula φ is a finite sequence of terms $(\varphi_1, \ldots, \varphi_\ell)$ such that $\varphi_\ell :\equiv \varphi$ and, for each $k \in \{1, \ldots, \ell\}$, the term φ_k is either

- $=t_1t_2$ for some terms t_1 and t_2
- $<t_1t_2$ for some terms t_1 and t_2
- $\neg\varphi_j$ for some $j < k$
- $(\varphi_i \vee \varphi_j)$ for some $i, j < k$
- $(\forall x)(\varphi_i)$ for some $i < k$ and $x \in \mathit{Vars}$

Construction Sequences for General Recursive Definitions

This idea is very general: using an appropriate notion of *construction sequence*, we get a Δ-definition of any recursively defined set or function.

For example, recall the Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

defined by $f(1) = f(2) = 1$ and $f(n) = f(n-1) + f(n-2)$ for $n \geq 3$. We can define the function $f(n)$ in terms of the set of codes of construction sequences

$$\textsc{FibonacciConSeq} = \{\langle f(1), f(2), \ldots, f(n)\rangle : n = 1, 2, \ldots\}.$$

Next Steps (Sections 5.11–5.12)

The following are Δ-definable:

$$\text{LogicalAxiom} := \{\ulcorner \varphi \urcorner : \varphi \text{ is a logical axiom}\}$$

$$\text{RuleOfInference} := \{(\langle \ulcorner \gamma_1 \urcorner, \ldots, \ulcorner \gamma_n \urcorner \rangle, \ulcorner \varphi \urcorner) : (\{\gamma_1, \ldots, \gamma_n\}, \varphi) \text{ is a rule of inference}\}$$

$$\text{Axiom}_N := \{\ulcorner N_1 \urcorner, \ldots, \ulcorner N_{11} \urcorner\}$$

$$\text{Deduction}_N := \{(\langle \ulcorner \delta_1 \urcorner, \ldots, \ulcorner \delta_1 \urcorner \rangle, \ulcorner \varphi \urcorner) : (\delta_1, \ldots, \delta_n) \text{ is a deduction from } N \text{ of } \varphi\}.$$

Important Δ-definable functions:

$$\text{Num}(a) := \ulcorner \overline{a} \urcorner,$$

$$\text{TermSub}(\ulcorner u \urcorner, \ulcorner x \urcorner, \ulcorner t \urcorner) := \ulcorner u_t^x \urcorner,$$

$$\text{Sub}(\ulcorner \varphi \urcorner, \ulcorner x \urcorner, \ulcorner t \urcorner) := \ulcorner \varphi_t^x \urcorner.$$