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SUCCESSOR-INVARIANT FIRST-ORDER LOGIC ON FINITE STRUCTURES

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Abstract. We consider successor-invariant first-order logic $(\mathrm{FO}+\mathrm{succ})_{\mathrm{inv}}$, consisting of sentences Φ involving an "auxiliary" binary relation S such that $(\mathfrak{A}, S_1) \models \Phi \iff (\mathfrak{A}, S_2) \models \Phi$ for all finite structures $\mathfrak{A}$ and successor relations S_1, S_2 on $\mathfrak{A}$. A successor-invariant sentence Φ has a well-defined semantics on finite structures $\mathfrak{A}$ with no given successor relation: one simply evaluates Φ on $(\mathfrak{A}, S)$ for an arbitrary choice of successor relation S. In this article, we prove that $(\mathrm{FO}+\mathrm{succ})_{\mathrm{inv}}$ is more expressive on finite structures than first-order logic without a successor relation. This extends similar results for order-invariant logic [7] and epsilon-invariant logic [10].

§1. Introduction. Let σ and τ be disjoint relational vocabularies, and let $\mathscr{C}$ be an isomorphism-closed class of τ-structures, which we call "auxiliary" structures. A sentence Φ in the first-order language of $\sigma \cup \tau$ is *$\mathscr{C}$-invariant* if $\mathfrak{A} \models \Phi \iff \mathfrak{B} \models \Phi$ for all $(\sigma\cup\tau)$-structures $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}|_\sigma = \mathfrak{B}|_\sigma$ and $\mathfrak{A}|_\tau, \mathfrak{B}|_\tau \in \mathscr{C}$ where $\mathfrak{A}|_\sigma$ denotes the σ-reduct of $\mathfrak{A}$ (and likewise: $\mathfrak{A}|_\tau, \mathfrak{B}|_\sigma, \mathfrak{B}|_\tau$). In other words, Φ is $\mathscr{C}$-invariant if it has the same semantics on every two $(\sigma\cup\tau)$-expansions of a σ-structure such that both τ-reducts are auxiliary structures in $\mathscr{C}$. A class $\mathscr{K}$ of σ-structures is *$\mathscr{C}$-invariantly definable* if there exists a $\mathscr{C}$-invariant sentence Φ such that $\mathscr{K} = \{\mathfrak{A}|_\sigma : \mathfrak{A} \models \Phi\}$, i.e., the class of σ-reducts of models of Φ.

$\mathscr{C}$-invariant definability is a trivial concept when the class $\mathscr{C}$ is first-order axiomatizable. For axiomatizable $\mathscr{C}$, Craig's Interpolation Theorem implies that every $\mathscr{C}$-invariantly definable class of σ-structures is definable in the first-order language of σ (without reference to an auxiliary τ-structure). However, as we will see, $\mathscr{C}$-invariance can be surprisingly non-trivial for non-axiomatizable classes $\mathscr{C}$, including many natural classes of finite structures.

Proviso 1.1. Henceforth, $\mathscr{C}$ will always be an isomorphism-closed class of finite τ-structures. We assume that $\mathscr{C}$ contains τ-structures of every finite cardinality (a property sometimes called having full finite spectrum). Moreover,

- *$\mathscr{C}$-invariant* means $\mathscr{C}$-invariant on finite structures,
- *definable* means first-order definable on finite structures, and
- following [9], we write $(\mathrm{FO}+\mathscr{C})_{\mathrm{inv}}$ for the set of $\mathscr{C}$-invariant sentences.

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One consequence of this proviso is that every finite set is the universe of some auxiliary τ-structure in $\mathscr{C}$. Therefore, every σ-structure has at least one $(\sigma \cup \tau)$-expansion whose τ-reduct belongs to $\mathscr{C}$. This observation leads to a well-defined semantics for sentences of $(\mathrm{FO} + \mathscr{C})_{\mathrm{inv}}$ on the class of finite σ-structures. For a $\mathscr{C}$-invariant sentence Φ and a finite σ-structure $\mathfrak{A}$, we define

$$\mathfrak{A} \models^{(\mathrm{FO}+\mathscr{C})_{\mathrm{inv}}} \Phi \iff \mathfrak{B} \models^{\mathrm{FO}} \Phi$$

where $\mathfrak{B}$ is any $(\sigma \cup \tau)$-expansion of $\mathfrak{A}$ such that $\mathfrak{B}|_\tau \in \mathscr{C}$. Equipped with this semantics, $(\mathrm{FO} + \mathscr{C})_{\mathrm{inv}}$ is called *$\mathscr{C}$-invariant logic*.

Remark 1.2. For many classes $\mathscr{C}$, $(\mathrm{FO} + \mathscr{C})_{\mathrm{inv}}$ fails to be a logic in the strict sense defined by Gurevich [8]. This is because the syntax of $(\mathrm{FO} + \mathscr{C})_{\mathrm{inv}}$, i.e. the set of $\mathscr{C}$-invariant sentences, is not recursive in general.

We now restrict our attention to a few specific auxiliary classes $\mathscr{C}$. One case of particular interest is when $\mathscr{C}$ is the class of finite linear orders; in this case, we speak of *order-invariance* and denote order-invariant logic by $(\mathrm{FO} + <)_{\mathrm{inv}}$. Similarly, when $\mathscr{C}$ is the class of finite successor relations, we speak of *successor-invariance* and denote successor-invariant logic by $(\mathrm{FO} + \mathrm{succ})_{\mathrm{inv}}$.

1.1. Order-invariant logic. Order-invariant logic arises naturally in settings such as database theory [1] and the theory of embedded finite models [9]. Logical properties of $(\mathrm{FO} + <)_{\mathrm{inv}}$ such as Gaifman locality have been studied in [6]. The first to study $(\mathrm{FO} + <)_{\mathrm{inv}}$ from the standpoint of finite model theory was Yuri Gurevich [7], to whom the following theorem is due.

Theorem 1.3. $(\mathrm{FO} + <)_{\mathrm{inv}}$ *is more expressive than* FO.

Proof. For a finite set X, whose elements we call "atoms", consider the *set-powerset structure* $\langle X \cup \wp(X), \in \rangle$ associated with X. Here $X \cup \wp(X)$ is the disjoint union of X and its powerset $\wp(X)$, and $\in$ is the usual set-membership relation (a subset of $X \times \wp(X)$). Let $\mathscr{G}$ be the isomorphism-closure of the class of finite set-powerset structures. It is an easy exercise to show that $\mathscr{G}$ is first-order definable. (To express in first-order language that $\wp(X)$ is the powerset of X, one says that the set of subsets of X represented by elements of the $\wp(X)$-sort contains all singletons and is closed under union. We spell out the same idea in a slightly different setting in the proof of Lemma 3.1.)

Now consider the subclass $\mathscr{G}_{\mathrm{even}}$ of $\mathscr{G}$-structures with an even number of atoms (i.e., where the X-sort has even cardinality). By playing Ehrenfeucht-Fraïssé games, one can show that any two set-powerset structures with at least 2^{r-1} atoms are indistinguishable by sentences of quantifier rank r. It follows that $\mathscr{G}_{\mathrm{even}}$ is not definable in first-order logic FO.

On the other hand, it turns out that $\mathscr{G}_{\mathrm{even}}$ is definable in $(\mathrm{FO} + <)_{\mathrm{inv}}$. The order-invariant sentence defining $\mathscr{G}_{\mathrm{even}}$ is the conjunction of a FO sentence defining $\mathscr{G}$ and a sentence asserting the existence of an element $y \in \wp(X)$ such that

- $x_{\min} \in y$ and $x_{\max} \notin y$ where $x_{\min}$ and $x_{\max}$ are respectively the minimal and maximal elements in X with respect to $<$, and
- $x_1 \in y \iff x_2 \notin y$ whenever x_1 immediately precedes x_2 in the linear order induced by $<$ on X.

There exists such an element $y \in \wp(X)$ if, and only if, X has even cardinality. Therefore, this sentence defines $\mathscr{G}_{\text{even}}$ in $(\text{FO} + <)_{\text{inv}}$. ⊣

The class $\mathscr{G}_{\text{even}}$ plays a critical role in the class $\mathscr{K}_{\text{even}}$ which we introduce in the next section in order to separate the logic $(\text{FO} + \text{succ})_{\text{inv}}$ from FO.

REMARK 1.4. Gurevich's class $\mathscr{G}_{\text{even}}$ is essentially identical to the class $\mathscr{BA}_{\text{even}}$ of finite Boolean algebras with an even number of atoms, which is similarly definable in $(\text{FO} + <)$ but not in FO (see Chapter 5 of [9] for a detailed proof).

1.2. Successor-invariant logic. Let us first of all clarify what we mean by "successor relation" in this paper.

TERMINOLOGY 1.5. Let X be a finite set. A *permutation* of X is a bijective function $p : X \longrightarrow X$. p is *circular* if all elements of X lie in a single orbit (i.e., each element of X is reachable from every other element via repeated application of p). A *successor relation* on X is the graph of a circular permutation. Abusing notation, given a successor relation S, we treat S both as a subset of $X \times X$ and at other times as the permutation it represents, as when we write $S(x)$ (respectively, $S^{-1}(x)$) for the unique y such that $(x, y) \in S$ (respectively, $(y, x) \in S$).

REMARK 1.6. There is another common notion of "successor relation" in the literature, what one might call a *linear* (as opposed to *circular*) successor relation, that is, the kind of successor relation induced by a linear order. Linear successor relations differ from circular successor relations in the existence of a unique *minimal* and *maximal* elements having, respectively, no predecessor and no successor.

The main result of this article, the separation of logics $\text{FO} \subsetneqq (\text{FO} + \text{succ})_{\text{inv}}$, holds true if we adopt the linear, rather than circular, definition of successor relation. This is because the two kinds of successor relation are interdefinable in the following sense. A linear successor relation is transformed into a circular successor relation simply by making the minimal element the successor of the maximal element. Conversely, a circular successor relation is made into a linear successor relation by choosing an arbitrary element (to become the maximum) and removing the succession-link between it and its successor. The choice of maximal element does not affect the semantics of sentences which are successor-invariant for linear successor relations.

The decision to work with circular successor relations is merely a matter of convenience. The choice between linear and circular might, however, have implications if we were concerned with syntactic questions about successor-invariant sentences (e.g., the minimal quantifier-rank required to define a successor-invariant property).

Successor-invariant logic arises naturally in many of the same settings as order-invariant logic. A salient example is the theory of relational databases. Real-world implementations often impose some additional structure (say, a successor relation) on the entries of a relation database. One would like to know whether this additional structure can be exploited by a first-order query language in a manner that does not depend on the particular implementation of the database. The question $(\text{FO} + \text{succ})_{\text{inv}} \stackrel{?}{=} \text{FO}$ thus arose in database theory [1] as well as in finite model theory. We resolve this question in the present article.

THEOREM 4.14. *Successor-invariant logic is more expressive than first-order logic.*

Our separating counterexample (a class of structures definable in $(\mathrm{FO}{+}\mathrm{succ})_{\mathrm{inv}}$ but not in FO) is a good deal more complicated than Gurevich's class $\mathscr{G}_{\mathrm{even}}$. We begin with the observation that the order-invariant sentence defining $\mathscr{G}_{\mathrm{even}}$ uses the auxiliary linear order on a set-powerset structure $\langle X \cup \wp(X), \in\rangle$ only insofar it induces a successor relation on the X-sort (the set of atoms); the restriction of $<$ to the $\wp(X)$-sort is unimportant. A linear order is of course indispensable for the purpose of inducing a successor relation on the X-sort. Were we instead given an auxiliary successor relation on $X \cup \wp(X)$, it is unclear how we might derive a successor relation on the set X alone. Indeed, it appears unlikely that that class $\mathscr{G}_{\mathrm{even}}$ is successor-invariantly definable, although this has not been proved.

The classes $\mathscr{K}$ and $\mathscr{K}_{\mathrm{even}}$ defined in the next section are built around $\mathscr{G}$ and $\mathscr{G}_{\mathrm{even}}$ with the intention of allowing us to derive a successor relation on a distinguished subset A given an auxiliary successor relation on the entire structure. The powerset of A is interpretable in structures of $\mathscr{K}$, so that once we derive a successor relation on A, its parity can be determined in the same manner as for set-powerset structures. The difficult part of the construction involves ensuring that the parity of A cannot be detected in the absence of an auxiliary successor relation on the full structure.

1.3. Epsilon-invariant logic. Epsilon-invariant logic $(\mathrm{FO}+\varepsilon)_{\mathrm{inv}}$ is a close cousin of the $\mathscr{C}$-invariant logics we have discussed so far. ε stands for Hilbert's epsilon "choice" operator, which selects a distinguished element from each nonempty subset of the universe of a structure. Formulas of epsilon-logic are first-order formulas with an additional term construct $\varepsilon_x\, \phi(x)$. Epsilon-invariant logic $(\mathrm{FO}+\varepsilon)_{\mathrm{inv}}$ consists of those sentences of epsilon-logic whose semantics does not depend on the particular choice of epsilon operator in any finite structure. The following result is due to Martin Otto [10].

THEOREM 1.7. *Epsilon-invariant logic is more expressive than first-order logic on finite structures.* ⊣

Order-invariant logic $(\mathrm{FO}+<)_{\mathrm{inv}}$ is obviously at least as expressive as epsilon-invariant logic $(\mathrm{FO}+\varepsilon)_{\mathrm{inv}}$. This follows easily from the fact that any finite linear order gives rise to an epsilon-operator by distinguishing the minimal element in each nonempty subset. However, it is unknown whether $(\mathrm{FO}+<)_{\mathrm{inv}}$ is more expressive as $(\mathrm{FO}+\varepsilon)_{\mathrm{inv}}$. The precise relationship between $(\mathrm{FO}+\mathrm{succ})_{\mathrm{inv}}$ and $(\mathrm{FO}+\varepsilon)_{\mathrm{inv}}$ is another interesting open question. We will revisit these questions in §5.

§2. Definition of the class $\mathscr{K}_{\mathrm{even}}$.

DEFINITION 2.1. Let Σ be a set of unary relation symbols, let $\Sigma_{\circledcirc}$ be a subset of Σ, and let $\sim$ be a binary relation symbol. A structure $\mathfrak{M}$ in a vocabulary containing $\Sigma \cup \{\sim\}$ is said to be $(\Sigma, \Sigma_{\circledcirc}, \sim)$-*sorted* if

- the universe of $\mathfrak{M}$ is a disjoint union of the sets (called *sorts*) defined by the relations in Σ, and

- $\sim$ defines an equivalence relation on each of the sorts in $\Sigma_{\circledcirc}$ and is vacuous elsewhere.

We now fix particular Σ and $\Sigma_{\circledcirc}$:

$$\Sigma = \{A, B, B^A_{\circledcirc}, C_{\circledcirc}, D, B^D, (B^D)^D, E, E^D_{\circledcirc}\},$$
$$\Sigma_{\circledcirc} = \{B^A_{\circledcirc}, C_{\circledcirc}, E^D_{\circledcirc}\}.$$

The appearance of unary relation symbols $B^A_{\circledcirc}, C_{\circledcirc}, B^D, (B^D)^D, E^D_{\circledcirc}$ is merely descriptive. Symbol B^D is intended to suggest the set of functions from sort D to sort B; however, this is not necessarily the case in an arbitrary $(\Sigma, \Sigma_{\circledcirc}, \sim)$-sorted structure. Of course, for the class of structures we are about to define, B^D will have exactly this interpretation.

We also fix a vocabulary σ defined by

$$\sigma = \Sigma \cup \{\sim, \blacktriangleright, \mathsf{app}, \mathsf{bar}, 0, 1\}$$

where $\sim$ and $\blacktriangleright$ are binary relation symbols, app and bar are ternary relations symbols, and 0 and 1 are constant symbols. When discussing $(\Sigma, \Sigma_{\circledcirc}, \sim)$-sorted σ-structures:

- A^{c} denotes the complement of sort A, and
- B^A, C, E^D respectively denote the quotient sets $B^A_{\circledcirc}/\!\sim$, $C_{\circledcirc}/\!\sim$, $E^D_{\circledcirc}/\!\sim$.

We now define classes $\mathscr{K}$ and $\mathscr{K}_{\mathrm{even}}$.

Definition 2.2. $\mathscr{K}$ is the class of finite $(\Sigma, \Sigma_{\circledcirc}, \sim)$-sorted σ-structures which satisfy the following five axioms.

(I) $B = \{0, 1\}$ *and* $0 \neq 1$.

(II) *Relation*

$$\blacktriangleright \subseteq (A \times C_{\circledcirc}) \cup (C_{\circledcirc} \times C_{\circledcirc}) \cup (C_{\circledcirc} \times D)$$

is a congruence[1] *with respect to* $\sim$ *such that, with respect to the induced binary relation*[2] $\rhd$ *on* $A \cup C \cup D$*:*
- *every element of* A *has in-degree 0 and out-degree 1;*
- *every element of* C *has in-degree* ≤ 1 *and out-degree 1;*
- *every element of* D *has in-degree 1 and out-degree 0.*

(III) *Relation*

$$\mathsf{app} \subseteq \bigcup_{\substack{X,Y,Z \in \Sigma \\ X \text{ is } Z^Y \text{ or } Z^Y_{\circledcirc}}} (X \times Y \times Z)$$

satisfies:
- *for every* $x \in X$*, the set* $\{(y, z) : (x, y, z) \in \mathsf{app}\}$ *is the graph of a function* $Y \longrightarrow Z$ (which we denote by x^*)*;*
- *for every function* $f : Y \longrightarrow Z$*, there exists* $x \in X$ *such that* $x^* = f$*;*
- *this* x *is unique if* X *is* Z^Y *and unique up to* $\sim$ *if* X *is* $Z^Y_{\circledcirc}$*.*

[1] i.e., for all $\gamma, \gamma' \in C_{\circledcirc}$, $\gamma \sim \gamma'$ implies $(\gamma \blacktriangleright x \iff \gamma' \blacktriangleright x)$ and $(x \blacktriangleright \gamma \iff x \blacktriangleright \gamma')$

[2] defined by $(a \rhd c \iff a \blacktriangleright \gamma)$ and $(c \rhd c' \iff \gamma \blacktriangleright \gamma')$ and $(c \rhd d \iff \gamma \blacktriangleright d)$ where $c, c' \in C$ are the $\sim$-equivalence classes of γ and γ', respectively

(IV) *Relation*

$$\mathsf{bar} \subseteq A^{\mathrm{c}} \times E \times B$$

has the property that for every $x \in A^{\mathrm{c}}$, the set $\{(e,b) : (x,e,b) \in \mathsf{bar}\}$ is the graph of a function $E \longrightarrow B$ (which we denote by $\overline{x}$).

(V) *For all distinct $x, y \in A^{\mathrm{c}}$, there exists $e \in E$ such that $\overline{x}(e) \neq \overline{y}(e)$.*

DEFINITION 2.3. $\mathscr{K}_{\mathrm{even}}$ is the subclass of $\mathscr{K}$-structures in which the sort A has even cardinality.

REMARK 2.4. $|A| \leq |D|$ in all $\mathscr{K}$-structures, as a consequence of axiom (II).

§3. $\mathscr{K}_{\mathrm{even}}$ is successor-invariantly definable. We begin by showing:

LEMMA 3.1. *$\mathscr{K}$ is first-order definable.*

PROOF. The property of being $(\Sigma, \Sigma_{\circledcirc}, \sim)$-sorted is obviously first-order definable, as are axioms (I), (II), (IV) and (V). This leaves only axiom (III). We render axiom (III) as a first-order sentence via an observation that exploits the finiteness of structures in $\mathscr{K}$. Consider any sorts $X, Y, Z \in \Sigma$ where X is Z^Y or $Z^Y_{\circledcirc}$. The only clause of axiom (III) that is non-trivial to write out as a first-order expression is the statement:

for every function $f : Y \longrightarrow Z$, there exists $x \in X$ such that $x^* = f$.

As we are dealing with finite structures, we can rephrase this statement in terms of a simple closure property:

for every $x_1 \in X$ and function $f : Y \longrightarrow Z$ that differs from x_1^* on at most one value of Y, there exists $x_2 \in X$ such that $x_2^* = f$.

Translated into first-order logic, this becomes:

$$\forall x_1 \in X\ \forall y_1 \in Y\ \forall z_1 \in Z\ \exists x_2 \in X$$
$$\underbrace{\mathsf{app}\, x_2 y_1 z_1}_{x_2^*(y_1)=z_1} \wedge \big(\forall y_2 \in Y\ y_2 \neq y_1 \rightarrow \underbrace{\forall z_2 \in Z\ \mathsf{app}\, x_2 y_2 z_2 \leftrightarrow \mathsf{app}\, x_1 y_2 z_2}_{x_2^*(y_2)=x_1^*(y_2)}\big).$$

One small detail: we must take care to add that X is nonempty whenever both Y and Z are nonempty. ⊣

We next establish an elementary combinatorial lemma.

LEMMA 3.2. *For any sets X and Y and distinct functions $f_1, \ldots, f_k : X \longrightarrow Y$, there exists a subset $X' \subseteq X$ of size $< k$ such that the restrictions $(f_1 \restriction X')$, $\ldots, (f_k \restriction X') : X' \longrightarrow Y$ are all distinct functions.*

PROOF. We construct X' inductively. Let $X_1 = \emptyset$. For $\ell < k$, suppose we have constructed $X_\ell \subseteq X$ of size $< \ell$ such that $(f_1 \restriction X_\ell), \ldots, (f_\ell \restriction X_\ell)$ are all distinct. If $(f_{\ell+1} \restriction X_\ell) \neq (f_j \restriction X_\ell)$ for all $j \in \{1, \ldots, \ell\}$, then we set $X_{\ell+1} = X_\ell$. Otherwise, there is a unique $j \in \{1, \ldots, \ell\}$ such that $(f_{\ell+1} \restriction X_\ell) = (f_j \restriction X_\ell)$. Since $f_{\ell+1} \neq f_j$, we can find $x \in X$ such that $f_{\ell+1}(x) \neq f_j(x)$. We now set $X_{\ell+1} = X_\ell \cup \{x\}$. Finally, having constructed $X_1, \ldots, X_k$, we set $X' = X_k$. ⊣

We now consider an arbitrary structure $\mathfrak{M}$ in the class $\mathscr{K}$. Let S be any successor relation on $\mathfrak{M}$. There exists a partition

$$A = P_1 \cup \cdots \cup P_k$$

(unique up to labeling of parts) with the property that for all $i \in \{1, \ldots, k\}$, there are unique elements $p_i, p_i' \in P_i$ such that

- $S(p_i), S^{-1}(p_i') \in A^{\mathrm{c}}$ and
- $S(q), S^{-1}(q) \in P_i$ for all $q \in P_i \setminus \{p_i, p_i'\}$.

In other words, P_i are the maximal S-paths inside A in which p_i is the final endpoint and p_i' is the origin.

$S(p_1), \ldots, S(p_k)$ are distinct elements of A^{c}. Therefore, $\overline{S(p_1)}, \ldots, \overline{S(p_k)}$ are distinct functions $E \longrightarrow B$ by axiom (V). So by Lemma 3.2, there is a set $E' \subseteq E$ with $|E'| < k$ such that $\overline{S(p_1)} \restriction E', \ldots, \overline{S(p_k)} \restriction E'$ are distinct functions $E' \longrightarrow B$. Since $|E'| < k \leq |A| \leq |D|$, there exists a function $f : D \longrightarrow E$ with $\mathrm{Range}(f) = E'$. Fix any choice of f.

By the same argument, there exists a function $f' : D \longrightarrow E$ such that $\overline{S^{-1}(p_1')} \restriction \mathrm{Range}(f'), \ldots, \overline{S^{-1}(p_k')} \restriction \mathrm{Range}(f')$ are all distinct functions $E \longrightarrow B$. Fix any choice of f'.

We next fix any surjective function h from D onto the set $\{1, \ldots, k\}$, and we define functions $g, g' : D \longrightarrow B^D$ by

$$\begin{aligned} g &: d \longmapsto \overline{S(p_{h(d)})} \circ f \\ g' &: d \longmapsto \overline{S^{-1}(p'_{h(d)+1})} \circ f' \end{aligned}$$

where the index $h(d) + 1$ is modulo k.

Finally, we select elements $F, F' \in E^D_{\circledcirc}$ and $G, G' \in (B^D)^D$ such that $F^* = f$, $F'^* = f'$, $G^* = g$ and $G'^* = g'$. Note that G and G' are uniquely determined, while F and F' are determined up to $\sim$ by axiom (III).

Definition 3.3. $\circledS(a, a', F, F', G, G')$ is the following first-order formula in vocabulary $\sigma \cup \{S\}$ with free variables a, a' ranging over sort A and parameters $F, F' \in E^D_{\circledcirc}$ and $G, G' \in (B^D)^D$ as chosen:

$$\begin{aligned} &\circledS(a, a', F, F', G, G') \stackrel{\text{def}}{=} \\ &\qquad \big(S(a) = a'\big) \vee \left(\begin{array}{l} \big(S(a) \in A^{\mathrm{c}}\big) \wedge \big(S^{-1}(a') \in A^{\mathrm{c}}\big) \wedge \\ \exists d \in D \ \big(\overline{S(a)} \circ f = g(d) \big) \wedge \big(\overline{S^{-1}(a')} \circ f' = g'(d) \big) \end{array} \right) \end{aligned}$$

In the above, the expression $\overline{S(a)} \circ f = g(d)$ abbreviates the formula

$$\forall \delta \in D \ \overline{S(a)}(f(\delta)) = (g(d))(\delta),$$

where $\overline{S(a)}(f(\delta)) = (g(d))(\delta)$ is itself a readable shorthand for

$$\exists e \in E \ \exists x \in A^{\mathrm{c}} \ \exists b \in B \ \exists H \in B^D \left(\begin{array}{l} \mathsf{app}\, F \delta e \wedge Sax \wedge \mathsf{bar}\, xeb \\ \wedge\, \mathsf{app}\, GdH \wedge \mathsf{app}\, H \delta b \end{array} \right)$$

The expression $\overline{S^{-1}(a')} \circ f' = g'(d)$ is similarly an abbreviation.

LEMMA 3.4. $Ⓢ(a, a', F, F', G, G')$ *defines a successor relation on* A*, namely the graph of the circular permutation*

$$a \longmapsto \begin{cases} S(a) & \text{if } S(a) \in A, \\ p'_{i+1} & \text{if } a = p_i. \end{cases}$$

PROOF. If $S(a) \in A$, then $S(a)$ is evidently the unique $a' \in A$ such that $(\mathfrak{M}, S) \models Ⓢ(a, a', F, F', G, G')$. This is immediate from the definition of Ⓢ.

It remains to show that for all $i \in \{1, \dots, k\}$, p'_{i+1} is the unique $a' \in A$ such that $(\mathfrak{M}, S) \models Ⓢ(p_i, a', F, F', G, G')$. This is a consequence of the way we chose parameters F, F', G, G'. Notice that any $a' \in A$ such that $\mathfrak{M} \models Ⓢ(p_i, a', F, F', G, G')$ must lie in the set $\{p'_1, \dots, p'_k\}$, since $S(p_i) \in A^{\mathrm{c}}$ forces $S^{-1}(a') \in A^{\mathrm{c}}$ (in order for $Ⓢ(p_i, a', F, F', G, G')$ to be satisfied) and $p'_1, \dots, p'_k$ are precisely the elements of A whose predecessors lie in A^{c}. By our selection of f and f', functions $\overline{S(p_1)} \circ f, \dots, \overline{S(p_k)} \circ f : D \longrightarrow B$ are all distinct, as are functions $\overline{S^{-1}(p'_1)} \circ f', \dots, \overline{S^{-1}(p'_k)} \circ f' : D \longrightarrow B$. By our choice of h and definition of g and g', we have

$$\overline{S(p_i)} \circ f = g(d) \iff \overline{S^{-1}(p'_{i+1})} \circ f' = g'(d) \iff h(d) = i.$$

Since h is a surjection from D onto $\{1, \dots, k\}$, there exists $d \in D$ such that $h(d) = i$ and thus $\overline{S(p_i)} \circ f = g(d)$ and $\overline{S^{-1}(p'_{i+1})} \circ f' = g'(d)$. It follows that $(\mathfrak{M}, S) \models Ⓢ(p_i, p'_{i+1}, F, F', G, G')$. Finally, if $(\mathfrak{M}, S) \models Ⓢ(p_i, p'_j, F, F', G, G')$ then we have $j = i + 1$. ⊣

DEFINITION 3.5. Φ is the following sentence in the first-order language of $\sigma \cup \{S\}$.

$$\begin{aligned} \Phi \stackrel{\text{def}}{=} \exists F, F' \in E^D_{\circledcirc}\ \exists G, G' \in (B^D)^D \\ \left(\begin{array}{l} \forall a' \in A\ \exists! a, a'' \in A \\ \qquad Ⓢ(a, a', F, F', G, G') \wedge Ⓢ(a', a'', F, F', G, G') \end{array} \right) \wedge \\ \left(\begin{array}{l} \exists \beta \in B^A_{\circledcirc}\ \forall a, a' \in A \\ \qquad Ⓢ(a, a', F, F', G, G') \rightarrow \underbrace{\mathsf{app}\,\beta a 0 \leftrightarrow \mathsf{app}\,\beta a' 1}_{\beta^*(a)=0 \iff \beta^*(a')=1} \end{array} \right) \end{aligned}$$

REMARK 3.6. In plain language, Φ says there is a choice of parameters F, F', G, G' such that

- Ⓢ defines the graph of a permutation of A (not necessarily circular!), and
- there exists a function $A \longrightarrow \{0, 1\}$ such that $a \longmapsto 0 \iff Ⓢ(a) \longmapsto 1$ for all $a \in A$.

LEMMA 3.7. *For every* $\mathfrak{M} \in \mathscr{K}$ *and successor relation* S *on* $\mathfrak{M}$*,*

$$(\mathfrak{M}, S) \models \Phi \iff \mathfrak{M} \in \mathscr{K}_{\mathrm{even}}.$$

PROOF. ($\Longrightarrow$) Suppose $(\mathfrak{M}, S) \models \Phi$. There is a choice for F, F', G, G' such that Ⓢ defines a permutation of A with the property that $a \longmapsto 0 \iff Ⓢ(a) \longmapsto 1$ for all $a \in A$. This implies that all Ⓢ-orbits in A have even cardinality. Therefore, A itself has even cardinality and so $\mathfrak{M} \in \mathscr{K}_{\mathrm{even}}$.

($\Longleftarrow$) Assume $\mathfrak{M} \in \mathscr{K}_{\text{even}}$. We demonstrate that $(\mathfrak{M}, S) \models \Phi$ by finding witnesses for the existential quantifiers in Φ. By Lemma 3.4, there exist F, F', G, G' such that Ⓢ defines a successor relation on A. So the clause

$$\forall a' \in A\ \exists! a, a'' \in A\ Ⓢ(a, a', F, F', G, G') \wedge Ⓢ(a', a'', F, F', G, G')$$

of Φ is clearly satisfied. It remains only to find a witness for β in the clause

$$\exists \beta \in B^A_{\circlearrowleft}\ \forall a, a' \in A\ Ⓢ(a, a', F, F', G, G') \to \big(\mathsf{app}\,\beta a 0 \leftrightarrow \mathsf{app}\,\beta a' 1\big).$$

This is easy, as A has even cardinality and Ⓢ is a successor relation on A. There are exactly two subsets $X \subseteq A$ with the property that $a \in X \iff Ⓢ(a) \notin X$. For either choice of X, take any $\beta \in B^A_{\circlearrowleft}$ such that $\beta^* : A \longrightarrow \{0, 1\}$ is the characteristic function of X. Such β clearly fits the bill. Thus, we conclude $(\mathfrak{M}, S) \models \Phi$. ⊣

THEOREM 3.8. *$\mathscr{K}_{\text{even}}$ is successor-invariantly definable.*

PROOF. $\mathscr{K}_{\text{even}}$ is defined in $(\text{FO} + \text{succ})_{\text{inv}}$ by the conjunction of Φ and the first-order axioms for $\mathscr{K}$. ⊣

REMARK 3.9. $\mathscr{K}_{\text{even}}$ is definable as well in epsilon-invariant logic $(\text{FO} + \varepsilon)_{\text{inv}}$. Formulas Ⓢ and Φ can be adapted to the scenario where, instead of an auxiliary successor relation on $\mathfrak{M}$, we are given an arbitrary injection $A \longrightarrow A^{\text{c}}$. An epsilon-operator ε on $\mathfrak{M}$ (i.e., a map associating each nonempty subset X of $\mathfrak{M}$ with a distinguished element $\varepsilon(X) \in X$) lets us define an injection $A \longrightarrow A^{\text{c}}$ via

$$a \longmapsto \varepsilon(\{\beta \in B^A_{\circlearrowleft} : (\forall a' \in A)\ \beta^*(a') = 1 \iff a' = a\}).$$

Since $\mathscr{K}_{\text{even}}$ is not first-order definable (as we show in the next section), it is thus a separating counterexample for the logics $\text{FO} \subsetneqq (\text{FO} + \varepsilon)_{\text{inv}}$. (The first separating counterexample for these logics is due to Otto [10].)

§4. $\mathscr{K}_{\text{even}}$ is not first-order definable. Our proof that $\mathscr{K}_{\text{even}}$ is undefinable consists of exhibiting, for every $r \in \mathbb{N}$, a pair of structures $\mathfrak{M} \in \mathscr{K}_{\text{even}}$ and $\mathfrak{N} \in \mathscr{K} \setminus \mathscr{K}_{\text{even}}$ such that $\mathfrak{M} \equiv_r \mathfrak{N}$. For the rest of this section, let r be a fixed natural number.

4.1. Probabilistic construction of $\mathfrak{M}$ and $\mathfrak{N}$. Let $\sigma^- = \sigma \setminus \{\mathsf{bar}\}$, and let $\mathscr{K}^-$ be the class of $(\Sigma, \Sigma_{\circlearrowleft}, \sim)$-sorted σ-structures which satisfy axioms (I)–(III) of Definition 2.2. The next lemma, which follows immediately from definitions, describes a special family of $\mathscr{K}^-$-structures.

LEMMA 4.1. *For all $n \in \mathbb{N}$, there is a unique $\mathscr{K}^-$-structure up to isomorphism which satisfies the following three conditions.*

1. *Sort E has size n.*
2. *All $\sim$-equivalence classes in $B^A_{\circlearrowleft} \cup C_{\circlearrowleft} \cup E^D_{\circlearrowleft}$ have size n.*
3. *The digraph $\langle A \cup C \cup D, \rhd \rangle$, defined in axiom* (II) *of Definition 2.2, is as depicted in Figure 1, below.* ⊣

As depicted in Figure 1, sorts A and D have sizes 2^r and $2^r + r$, respectively. 2^r elements of D are connected to the different elements of A via paths of length $2^r + 1$ which extend through C. The other r elements of D are the terminal endpoints of paths of length 2^r which originate in C.

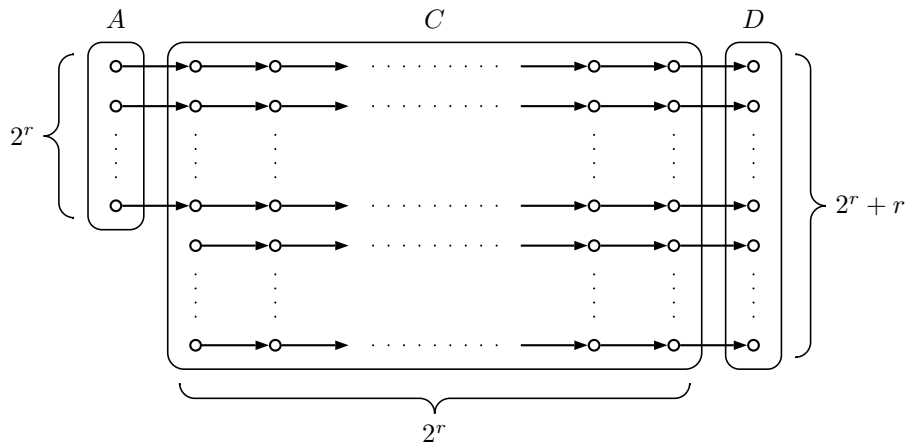

FIGURE 1. The digraph $\langle A \cup C \cup D, \rhd \rangle$

Let $\mathscr{K}'$ be the class of finite $(\Sigma, \Sigma_{\circledcirc}, \sim)$-sorted σ-structures which satisfy axioms (I)–(IV) of Definition 2.2 (but not necessarily axiom (V)). $\mathscr{K}'$-structures are essentially $\mathscr{K}$-structures where we do not insist that functions $\bar{x}, \bar{y} : E \longrightarrow B$ be different for distinct $x, y \in A^{\mathrm{c}}$.

DEFINITION 4.2. For all $n \in \mathbb{N}$, let $\mathfrak{M}_n^-$ be a fixed $\mathscr{K}^-$-structure satisfying conditions (1)–(3) of Lemma 4.1, and let $\mathfrak{M}_n$ be a <u>random</u> $\mathscr{K}'$-structure where

- $\mathfrak{M}_n^-$ is the σ^--reduct of $\mathfrak{M}_n$ (with probability 1), and
- relation $\mathsf{bar} \subseteq A^{\mathrm{c}} \times E \times B$ is defined probabilistically such that

$$\Pr\left[\overline{x}(e) = 1\right] = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{if } x \in E \setminus \{e\}, \\ \frac{1}{2} & \text{if } x \in A^{\mathrm{c}} \setminus E, \end{cases}$$

  independently for all $x \in A^{\mathrm{c}}$ and $e \in E$.

We now show that a certain property of $\mathscr{K}'$-structures called ℓ-extendibility holds for $\mathfrak{M}_n$ with limit probability 1 as $n \to \infty$. The notion of ℓ-extendibility and the method for proving $\mathfrak{M}_n$ is almost surely ℓ-extendibility are closely related to the notion of ℓ-extendibility for graphs and well-known probabilistic arguments for the existence of finite ℓ-extendible graphs (see e.g. [2]).

DEFINITION 4.3. A $\mathscr{K}'$-structure is *ℓ-extendible* ($\ell \in \mathbb{N}$) if the following two conditions hold.

- $\circledcirc_\ell$ For every set $X \subseteq A^{\mathrm{c}} \setminus E$ of size $\leq \ell$ and every function $f : X \longrightarrow B$, there are at least ℓ different $e \in E$ such that $\overline{x}(e) = f(x)$ for all $x \in X$.
- $\circledast_\ell$ For every $\sim$-equivalence class $Y \subseteq B^A_{\circledcirc} \cup C_{\circledcirc} \cup E^D_{\circledcirc}$, every set $Z \subseteq E$ of size $\leq \ell$ and every function $g : Z \longrightarrow B$, there are at least ℓ different $y \in Y$ such that $\overline{y}(z) = g(z)$ for all $z \in Z$.

REMARK 4.4. Every (≥ 2)-extendible $\mathscr{K}'$-structure is a $\mathscr{K}$-structure, as condition $\circledcirc_\ell$ implies axiom (V) for all $\ell \geq 2$. Recall axiom (V) states that for all

distinct $x, y \in A^{\mathrm{c}}$, there exists $e \in E$ such that $\overline{x}(e) \neq \overline{y}(e)$. Condition $\circledcirc_\ell$ with $X = \{x, y\}$ and $f : X \longrightarrow B$ defined by $f(x) = 0$ and $f(y) = 1$ guarantees that there exist (at least ℓ different) $e \in E$ such that $\overline{x}(e) = f(x) = 0$ and $\overline{y}(e) = f(y) = 1$.

LEMMA 4.5. *For all* $\ell \in \mathbb{N}$, $\lim_{n\to\infty} \Pr\big[\mathfrak{M}_n \text{ is } \ell\text{-extendible}\big] = 1$.

PROOF. Fix $\ell \in \mathbb{N}$. We claim that there exist constants $\lambda > 0$ and $0 < \mu < 1$ such that statements (a)–(d) hold for all large enough n.

(a) There are $\leq n^\lambda$ different pairs (X, f) where X is a set of $\leq \ell$ elements of $A^{\mathrm{c}} \setminus E$ and f is a function $X \longrightarrow B$.

(b) For each pair (X, f), the probability that (X, f) witnesses the failure of $\circledcirc_\ell$ is $\leq \mu^n$.

(c) There are $\leq n^\lambda$ different triples (Y, Z, g) where Y is a $\sim$-equivalence class in $B^A_{\circledcirc} \cup C_{\circledcirc} \cup E^D_{\circledcirc}$, Z is a set of $\leq \ell$ elements of E and g is a function $Z \longrightarrow B$.

(d) For each triple (Y, Z, g), the probability that (Y, Z, g) witnesses the failure of $\circledast_\ell$ is $\leq \mu^n$.

Assume for the moment that (a)–(d) hold for some μ and λ. Then for all large enough n,

$$\begin{aligned}
\Pr\big[\mathfrak{M}_n \text{ is not } \ell\text{-extendible}\big] &\leq \Pr\big[\mathfrak{M}_n \text{ fails } \circledcirc_\ell\big] + \Pr\big[\mathfrak{M}_n \text{ fails } \circledast_\ell\big] \\
&= \Pr\big[\exists (X, f) \text{ witnessing } \neg\circledcirc_\ell\big] + \Pr\big[\exists (Y, Z, g) \text{ witnessing } \neg\circledast_\ell\big] \\
&\leq \sum_{X,f} \Pr\big[(X, f) \text{ witnesses } \neg\circledcirc_\ell\big] + \sum_{Y,Z,g} \Pr\big[(Y, Z, g) \text{ witnesses } \neg\circledast_\ell\big] \\
&\leq 2n^\lambda \mu^n.
\end{aligned}$$

Since $2n^\lambda\mu^n \to 0$ as $n \to \infty$, it follows that $\lim_{n\to\infty} \Pr\big[\mathfrak{M}_n \text{ is not } \ell\text{-extendible}\big] = 0$ and, therefore, $\lim_{n\to\infty} \Pr\big[\mathfrak{M}_n \text{ is } \ell\text{-extendible}\big] = 1$.

It remains to show that there exist λ and μ which satisfy (a)–(d). We find λ which meets conditions (a) and (c) by counting:

$$\begin{aligned}
&\#Z \leq |E|^\ell = n^\ell = n^{\emptyset(1)} \\
&\#Y = |B^A| + |C| + |E^D| = 2^{2^r} + 2^r{\cdot}(2^r + r) + n^{2^r+r} < n^{\emptyset(1)} \\
&\#X \leq |A^{\mathrm{c}} \setminus E|^\ell = \Big(\underbrace{|B|+|D|+|B^D|+|(B^D)^D|}_{\emptyset(1)} + \underbrace{|B^A_{\circledcirc}|+|C_{\circledcirc}|+|E^D_{\circledcirc}|}_{n\cdot(|B^A|+|C|+|E^D|)}\Big)^\ell < n^{\emptyset(1)}
\end{aligned}$$

We also count:

$$\big(\#f \text{ for any given } X\big) = \big(\#g \text{ for any given } Z\big) \leq 2^\ell = \emptyset(1).$$

It follows that $\#(X, f) < n^{\emptyset(1)}$ and $\#(Y, Z, g) < n^{\emptyset(1)}$. Therefore, there exists a constant λ as in the hypothesis. (Any $\lambda > \ell{\cdot}(2^r + r + 1)$ does the job.)

Now consider a particular pair (X, f) and let $\ell_0 = |X|$. For all $e \in E$, we have

$$\Pr\big[\forall x \in X\ \overline{x}(e) = f(x)\big] = 2^{-\ell_0}.$$

Since events $\exists x \in X\ \overline{x}(e) \neq f(x)$ are independent as e ranges over E, we have

$$\begin{aligned}\Pr\left[(X,f) \text{ witnesses } \neg\circledcirc_\ell\right] &= \Pr\left[\left|\{e \in E : \forall x \in X\ \overline{x}(e) = f(x)\}\right| < \ell\right] \\ &= \sum_{j=0}^{\ell-1} \Pr\left[\left|\{e \in E : \forall x \in X\ \overline{x}(e) = f(x)\}\right| = j\right] \\ &< \sum_{j=0}^{\ell-1} \binom{n}{j} (2^{-\ell_0})^j (1-2^{-\ell_0})^{n-j} \\ &< \ell n^\ell (1-2^{-\ell})^{n-\ell}.\end{aligned}$$

We derive the last inequality using $\binom{n}{j} \leq n^\ell$ and $(2^{-\ell_0})^j \leq 1$ and $(1-2^{-\ell_0})^{n-j} \leq (1-2^{-\ell})^{n-\ell}$, as $0 \leq j, \ell_0 \leq \ell$. For any μ such that $1-2^{-\ell} < \mu < 1$, we see that $\ell n^\ell (1-2^{-\ell})^{n-\ell} < \mu^n$ for all large enough n.

A similar argument shows that $\Pr\left[(Y,Z,g) \text{ witnesses } \neg\circledast_\ell\right]$ is also eventually bounded by μ^n for some choice of $0 < \mu < 1$. ⊣

Definition 4.6. Structures $\mathfrak{M}$ and $\mathfrak{N}$ are defined as follows.

(1) Let ℓ_r equal $2^{4^{r+1}}$ and choose n sufficiently large such that $\Pr\left[\mathfrak{M}_n \text{ is } \ell_r\text{-extendible}\right] > 0$.
(2) Let $\mathfrak{M}$ be a fixed ℓ_r-extendible structure from the distribution associated with the random structure $\mathfrak{M}_n$.
(3) For all $\beta \in B^A_{\circledcirc}$, let $\beta^\dagger$ denote the set $\{a \in A : \beta^*(a) = 1\}$.
(4) Let a_0 be a fixed element of A, and let $\mathfrak{N}$ be the induced substructure of $\mathfrak{M}$ after removing a_0 as well as all $\beta \in B^A_{\circledcirc}$ such that $a_0 \in \beta^\dagger$.
(5) We denote by A' the set $A \setminus \{a_0\} = A^{\mathfrak{N}}$ (i.e., the interpretation of sort A in $\mathfrak{N}$) and by $B^{A'}_{\circledcirc}$ the set $\{\beta \in B^A_{\circledcirc} : a_0 \notin \beta^\dagger\} = (B^A_{\circledcirc})^{\mathfrak{N}}$ (i.e., the interpretation of sort $B^A_{\circledcirc}$ in $\mathfrak{N}$). All other sorts are the same in $\mathfrak{N}$ as in $\mathfrak{M}$.

Clearly, $\mathfrak{M} \in \mathscr{K}_{\text{even}}$ and $\mathfrak{N} \in \mathscr{K} \setminus \mathscr{K}_{\text{even}}$. Also note that $\mathfrak{N}$ is ℓ_r-extendible. This follows from the fact that $\mathfrak{N}$ is obtained from an ℓ_r-extendible structure (namely $\mathfrak{M}$) by purging individual element(s) from sort A and entire $\sim$-equivalence class(es) from sort $B^A_{\circledcirc}$. Looking closely at Definition 4.3 of ℓ-extendibility, one sees that conditions $\circledcirc_\ell$ and $\circledast_\ell$ are preserved under the act of purging sorts A and $B^A_{\circledcirc}$ in this manner.

4.2. Secondary structures related to $\mathfrak{M}$ and $\mathfrak{N}$. It is convenient at this juncture to define a couple of secondary structures derived from $\mathfrak{M}$ and $\mathfrak{N}$:

$$\begin{aligned}\mathfrak{M}^\dagger &= \langle A \cup \wp(A), \in\rangle, & \mathfrak{N}^\dagger &= \langle A' \cup \wp(A'), \in\rangle, \\ \mathfrak{M}^\ddagger &= \langle A \cup C \cup D, \rhd\rangle, & \mathfrak{N}^\ddagger &= \langle A' \cup C \cup D, \rhd\rangle.\end{aligned}$$

$\mathfrak{M}^\dagger$ and $\mathfrak{N}^\dagger$ are the familiar set-powerset structures we saw in our discussion of the class $\mathscr{G}$ in §1.1. $\mathfrak{M}^\ddagger$ is precisely the directed graph depicted in Figure 1. $\mathfrak{N}^\ddagger$ is the subgraph of $\mathfrak{M}^\ddagger$ after removing the element a_0. Note that structures $\mathfrak{M}^\dagger$ and $\mathfrak{M}^\ddagger$ are interpretable in $\mathfrak{M}$, and likewise $\mathfrak{N}^\dagger$ and $\mathfrak{N}^\ddagger$ in $\mathfrak{N}$.

Every partial isomorphism $\pi : \mathfrak{M} \longrightarrow \mathfrak{N}$ induces partial isomorphisms $\pi^\dagger : \mathfrak{M}^\dagger \longrightarrow \mathfrak{N}^\dagger$ and $\pi^\ddagger : \mathfrak{M}^\ddagger \longrightarrow \mathfrak{N}^\ddagger$ defined by

$$\pi^\dagger = \big(A \times A' \cap \pi\big) \cup \big\{(\beta^\dagger, \beta'^\dagger) : (\beta, \beta') \in B^A_{\circledcirc} \times B^{A'}_{\circledcirc} \cap \pi\big\},$$
$$\pi^\ddagger = \big(A \times A' \cap \pi\big) \cup \big\{(\gamma/\!\sim, \gamma'/\!\sim) : (\gamma, \gamma') \in C_{\circledcirc} \times C_{\circledcirc} \cap \pi\big\} \cup \big\{(d, d) : d \in D\big\}$$

where $\gamma/\!\sim, \gamma'/\!\sim \in C$ are the $\sim$-equivalence classes of γ and γ', respectively. The induced partial isomorphisms $\pi^\dagger$ and $\pi^\ddagger$ on these secondary structures facilitate our description of a family of "good" partial isomorphisms $\mathfrak{M} \longrightarrow \mathfrak{N}$ in the next subsection.

An additional bit of notation: for an element x in $\mathfrak{M}^\ddagger$ or $\mathfrak{N}^\ddagger$, let $\rhd^j(x)$ (respectively, $\rhd^{-j}(x)$) denote the unique jth $\rhd$-successor (respectively, jth $\rhd$-predecessor) of x, when defined (see Figure 1). When we speak of the $\rhd$-*distance* between elements x and y (respectively, between an element x and a set Y) in $\mathfrak{M}^\ddagger$ or $\mathfrak{N}^\ddagger$, we mean the minimal $j \in \mathbb{N}$ such that $\rhd^{\pm j}(x) = y$ (respectively, $\rhd^{\pm j}(x) \in Y$).

4.3. k-good partial isomorphisms $\mathfrak{M} \longrightarrow \mathfrak{N}$. We begin with a familiar definition.

DEFINITION 4.7. Let $\mathfrak{A}$ and $\mathfrak{B}$ be structures in the same vocabulary. A partial isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ defined by $x_1, \ldots, x_m \longmapsto y_1, \ldots, y_m$ (where x_i are elements of $\mathfrak{A}$ and y_i are elements of $\mathfrak{B}$) is a *j-equivalence* if $(\mathfrak{A}, x_1, \ldots, x_m) \equiv_j (\mathfrak{B}, y_1, \ldots, y_m)$, that is,

$$\mathfrak{A} \models \Psi(x_1, \ldots, x_m) \iff \mathfrak{B} \models \Psi(y_1, \ldots, y_m)$$

for all m-ary first-order formulas Ψ of quantifier rank at most j.

j-equivalences have the following "back and forth" property. (Lemma 4.8, below, can be taken for an inductive definition of j-equivalence where, in the base case, 0-equivalences are just partial isomorphisms.)

LEMMA 4.8. *If $\pi : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a j-equivalence where $j \geq 1$, then for every $x \in \mathfrak{A}$ (respectively, $y \in \mathfrak{B}$), there exists $y \in \mathfrak{B}$ (respectively, $x \in \mathfrak{A}$) such that $\pi \cup \{(x, y)\} : \mathfrak{A} \longrightarrow \mathfrak{B}$ is a $(j-1)$-equivalence.* ⊣

The class of k-good partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$ is a special subclass of $(r-k)$-equivalences with a similar "back and forth" property (Lemma 4.11). From the standpoint of Ehrenfeucht-Fraïssé games, k-goodness provides the criterion for a winning strategy in the k-round game on structures $\mathfrak{M}$ and $\mathfrak{N}$.

DEFINITION 4.9. For $k \in \{0, \ldots, r\}$, a partial isomorphism $\pi : \mathfrak{M} \longrightarrow \mathfrak{N}$ is *k-good* if it satisfies conditions (i)–(v), below.

(i) $\mathrm{Dom}(\pi)$ *contains* $B \cup D \cup B^D \cup (B^D)^D$ *and* π *acts as the identity on this set, that is,* $\pi(x) = x$ *for all* $x \in B \cup D \cup B^D \cup (B^D)^D$.

(ii) $f^*(d) \in \mathrm{Dom}(\pi)$ *for all* $f \in E^D_{\circledcirc} \cap \mathrm{Dom}(\pi)$ *and* $d \in D$.

(iii) *Induced partial isomorphism* $\pi^\dagger : \mathfrak{M}^\dagger \longrightarrow \mathfrak{N}^\dagger$ *is an* $(r-k)$*-equivalence.*

(iv) *Induced partial isomorphism* $\pi^\ddagger : \mathfrak{M}^\ddagger \longrightarrow \mathfrak{N}^\ddagger$ *satisfies the following for all* $c \in C \cap \mathrm{Dom}(\pi^\ddagger)$*:*

- c *and* $\pi^\ddagger(c)$ *have the same* $\rhd$*-distance to the set* D,

- *if* $\triangleright^{\pm j}(c) \in \mathrm{Dom}(\pi^\ddagger)$ *where* $j < 2^{r-k}$, *then* $\pi^\ddagger(\triangleright^{\pm j}(c)) = \triangleright^{\pm j}(\pi^\ddagger(c))$,
- *if* $\triangleright^{-j}(c) \in A$ *where* $j < 2^{r-k}$, *then* $\triangleright^{-j}(c) \in \mathrm{Dom}(\pi^\ddagger)$.

(v) *The following inequalities hold:*

- $|(B^A_{\circledcirc} \cup C_{\circledcirc} \cup E^D_{\circledcirc}) \cap \mathrm{Dom}(\pi)| \leq k$,
- $|E \cap \mathrm{Dom}(\pi)| \leq k|D|$,
- $|(A^{\mathrm{c}} \setminus E) \cap \mathrm{Dom}(\pi)| < \ell_r$.

Condition (i) says that all k-good partial isomorphisms extend the identity function on the common subset $B \cup D \cup B^D \cup (B^D)^D$ of $\mathfrak{M}$ and $\mathfrak{N}$.

Condition (ii) gives a closure property of k-good π. It is equivalently stated as $\pi(f^*(d)) = (\pi(f))^*(d)$ for all $f \in E^D_{\circledcirc} \cap \mathrm{Dom}(\pi)$ and $d \in D$.

Condition (iii) is a bridge to a well-known result about j-equivalence of set-powerset structures. This saves us from having to reinvent the wheel in the course of proving Lemma 4.11.

Condition (iv) implies that $\pi^\ddagger : \mathfrak{M}^\ddagger \longrightarrow \mathfrak{N}^\ddagger$ is an $(r-k)$-equivalence. It also gives a closure property on π, requiring that an element $a \in A$ belongs to the domain (respectively, range) of π whenever it is within $\blacktriangleright$-distance $2^{r-k} - 1$ of an element of $C_{\circledcirc} \cap \mathrm{Dom}(\pi)$ (respectively, $C_{\circledcirc} \cap \mathrm{Range}(\pi)$).

Condition (v) bounds the size of π over various sorts. Note that the first inequality implies the third, as

$$|(A^{\mathrm{c}} \setminus E) \cap \mathrm{Dom}(\pi)| = \underbrace{|B \cup D \cup B^D \cup (B^D)^D|}_{2+(2^r+r)+2^{(2^r+r)}+2^{(2^r+r)^2}} + \underbrace{|(B^A_{\circledcirc} \cup C_{\circledcirc} \cup E^D_{\circledcirc}) \cap \mathrm{Dom}(\pi)|}_{\leq r} < 2^{4^{r+1}} = \ell_r.$$

Also note that we can replace $\mathrm{Dom}(\pi)$ with $\mathrm{Range}(\pi)$ in any of these inequalities, since π is a partial isomorphism and therefore preserves sorts.

LEMMA 4.10. *The identity function on the set* $B \cup D \cup B^D \cup (B^D)^D$ *is a* 0*-good partial isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$.

PROOF. Let π_0 denote the identity function on $B \cup D \cup B^D \cup (B^D)^D$. π_0 trivially satisfies conditions (i), (ii), (iv) and (v) of 0-goodness. As for condition (iii), notice that $\pi_0^\dagger$ is the empty partial isomorphism from $\mathfrak{M}^\dagger$ to $\mathfrak{N}^\dagger$. So condition (iii) is equivalent to the statement that $\mathfrak{M}^\dagger$ and $\mathfrak{N}^\dagger$ are r-equivalent, i.e., that they satisfy the same first-order sentences of quantifier rank r. It is well-known that every two set-powerset structures with at least 2^{r-1} atoms are indistinguishable by sentences of quantifier rank r (see proof of Theorem 1.3). Since $\mathfrak{M}^\dagger$ and $\mathfrak{N}^\dagger$ are set-powerset structures with 2^r and $2^r - 1$ atoms, respectively, it follows that $\pi_0^\dagger$ is an r-equivalence and, therefore, that π_0 is 0-good. ⊣

LEMMA 4.11. *Suppose* π *is a* k*-good partial isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$ *where* $k < r$. *Then every* $x \in \mathfrak{M}$ (*respectively,* $y \in \mathfrak{N}$) *belongs to the domain* (*respectively, range*) *of a* $(k+1)$*-good partial isomorphism* $\widehat{\pi} : \mathfrak{M} \longrightarrow \mathfrak{N}$ *extending* π.

PROOF. Let x be any element of $\mathfrak{M}$. We prove that π has a $(k+1)$-good extension whose domain contains x. By symmetry of the argument in $\mathfrak{M}$ and $\mathfrak{N}$, we get for free the other direction of the lemma: for every $y \in \mathfrak{N}$, there is a $(k+1)$-good extension of π whose range contains y.

We assume that $x \notin \mathrm{Dom}(\pi)$, since in the event that $x \in \mathrm{Dom}(\pi)$ we simply let $\widehat{\pi} = \pi$ (as a k-good partial isomorphism is clearly also $(k+1)$-good). We consider five cases, according to whether x belongs to sort A, $B^A_{\circledcirc}$, $C_{\circledcirc}$, E or $E^D_{\circledcirc}$. Note that x cannot belong to B, D, B^D or $(B^D)^D$, because of the assumption that $x \notin \mathrm{Dom}(\pi)$. In each of the five cases below, we explain how $\widehat{\pi}$ is defined and leave verification that $\widehat{\pi}$ is $(k+1)$-good as a straightforward exercise.

Case 1: $x \in A$

Since $\pi^\dagger : \mathfrak{M}^\dagger \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{N}^\dagger$ is an $(r-k)$-equivalence by condition (iii), there exists $x' \in A'$ such that $\pi^\dagger \cup \{(x, x')\}$ is an $(r-k-1)$-equivalence by Lemma 4.8. The partial isomorphism $\widehat{\pi} : \mathfrak{M} \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{N}$ is now defined by $\widehat{\pi} = \pi \cup \{(x, x')\}$.

Case 2: $x \in B^A_{\circledcirc}$

Let $y = x^\dagger \in \wp(A)$. Again using the fact that $\pi^\dagger : \mathfrak{M}^\dagger \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{N}^\dagger$ is an $(r-k)$-equivalence, there exists $y' \in \wp(A')$ such that $\pi^\dagger \cup \{(y, y')\}$ is an $(r-k-1)$-equivalence.

We now invoke the ℓ_r-extendibility of $\mathfrak{N}$ (property $\circledast$) to find ℓ_r distinct $x' \in B^{A'}_{\circledcirc}$ such that $x'^\dagger = y'$ and $\overline{x'}(e) = \overline{x}(\pi^{-1}(e))$ for all $e \in E \cap \mathrm{Range}(\pi)$. Since $|\mathrm{Range}(\pi)| < \ell_r$ by condition (v), by the pigeonhole principle at least one of these x' does not belong to $\mathrm{Range}(\pi)$. Choose any such x' and let $\widehat{\pi} = \pi \cup \{(x, x')\}$.

Case 3: $x \in C_{\circledcirc}$

Let $c \in C$ be the $\sim$-equivalence class of x. We begin by finding a suitable $c' \in C$ from which to pick the image x' of x under $\widehat{\pi}$. First, assume

$$(\star) \qquad \rhd^{\pm j}(c) \in \mathrm{Dom}(\pi^\ddagger) \text{ for some } j < 2^{r-k-1}.$$

In this case, let $c' = \rhd^{\mp j}(\pi^\ddagger(\rhd^{\pm j}(c)))$. Due to condition (iv), c' does not depend on the value of j, if more than one choice is possible.

Next suppose $(\star)$ does not hold, and let us instead assume

$$(\star\star) \qquad \rhd^{-j}(c) \in A \text{ for some } j < 2^{r-k-1}.$$

Let $a = \rhd^{-j}(c)$ and, proceeding as in Case 1, we find $a' \in A'$ such that $\pi^\dagger \cup \{(a, a')\}$ is an $(r-k-1)$-equivalence between $\mathfrak{M}^\dagger$ and $\mathfrak{N}^\dagger$. Now let $c' = \rhd^j(a')$.

Finally, suppose neither $(\star)$ nor $(\star\star)$ holds. Let $c'_1, \ldots, c'_r$ be distinct elements of C such that in the directed graph $\mathfrak{N}^\ddagger$,

- c'_i has the same $\rhd$-distance to D as c, and
- c'_i is not $\rhd$-connected to A.

Since $|C_{\circledcirc} \cap \mathrm{Range}(\pi)| \leq k < r$ by condition (v), there is some c'_i which is not $\rhd$-connected to the set $C \cap \mathrm{Range}(\pi^\ddagger)$. Let $c' = c'_i$.

Now that $c' \in C$ has been chosen in all cases, proceeding as in Case 2, we invoke the ℓ_r-extendibility of $\mathfrak{N}$ (property $\circledast$) to find ℓ_r distinct $x' \in C_{\circledcirc}$ such that c' is the $\sim$-equivalence class of x' and $\overline{x'}(e) = \overline{x}(\pi^{-1}(e))$ for all $e \in E \cap \mathrm{Range}(\pi)$. Since $|\mathrm{Range}(\pi)| < \ell_r$, at least one of these x' is not contained in $\mathrm{Range}(\pi)$. Choose any such x' and let

$$\widehat{\pi} = \begin{cases} \pi \cup \{(x, x'), (a, a')\} & \text{in case } (\star\star), \\ \pi \cup \{(x, x')\} & \text{otherwise.} \end{cases}$$

CASE 4: $x \in E$

We prove a stronger claim (which will help us in CASE 5). Let $e_1, \dots, e_j$ be any elements of E where $j \leq |D|$. We wish to show:

$(\star\star\star)$ *There exist* $e'_1, \dots, e'_j \in E$ *such that* $\widetilde{\pi} = \pi \cup \{(e_1, e'_1), \dots, (e_j, e'_j)\}$ *is a* $(k+1)$*-good partial isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$.

We proceed by induction. For $i \in \{1, \dots, j\}$, suppose we have found $e'_1, \dots, e'_{i-1}$ such that $\widetilde{\pi}_{i-1} = \pi \cup \{(e_1, e'_1), \dots, (e_{i-1}, e'_{i-1})\}$ is a $(k+1)$-good partial isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$. We choose e'_i as follows. If $e_i \in \mathrm{Dom}(\widetilde{\pi}_{i-1})$, then let $e'_i = \widetilde{\pi}_{i-1}(e_i)$. Otherwise, we use the ℓ_r-extendibility of $\mathfrak{N}$ (this time property $\circledcirc$) to come up with ℓ_r different $e' \in E$ such that $\overline{x'}(e') = \overline{x'}(e_i)$ for all $x' \in (A^{\mathrm{c}} \setminus E) \cap \mathrm{Range}(\pi)$. Since $|(E \cup \{e'_1, \dots, e'_{i-1}\}) \cap \mathrm{Range}(\pi)| < r|D| < \ell_r$, at least one of these e' is not contained in $\mathrm{Range}(\pi)$. Let e'_i be any such e'. It is straightforward to check that $\widetilde{\pi}_i = \widetilde{\pi}_{i-1} \cup \{(e_i, e'_i)\}$ is $(k+1)$-good. The claim $(\star\star\star)$ follows by induction.

CASE 5: $x \in E^D_{\circledcirc}$

Let $e_1, \dots, e_j$ enumerate the set $\{x^*(d) : d \in D\}$. By the claim $(\star\star\star)$ of CASE 4, there exist $e'_1, \dots, e'_j \in E$ such that $\widetilde{\pi} = \pi \cup \{(e_1, e'_1), \dots, (e_j, e'_j)\}$ is a $(k+1)$-good partial isomorphism $\mathfrak{M} \longrightarrow \mathfrak{N}$.

By the ℓ_r-extendibility of $\mathfrak{N}$ (property $\circledast$), there exist ℓ_r distinct $x' \in E^D_{\circledcirc}$ such that $x'^* = \widetilde{\pi} \circ x^*$ and $\overline{x'}(e) = \overline{x}(\widetilde{\pi}^{-1}(e))$ for all $e \in E \cap \mathrm{Range}(\widetilde{\pi})$. Since $|\mathrm{Range}(\pi)| < \ell_r$, at least one of these x' is not in $\mathrm{Range}(\pi)$. Choose any such x' and let $\widehat{\pi} = \widetilde{\pi} \cup \{(x, x')\}$. ⊣

COROLLARY 4.12. *For* $k \in \{0, \dots, r\}$*, every partial isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{N}$ *which admits a* k*-good extension is an* $(r-k)$*-equivalence.*

PROOF. k-goodness provides the basis for a winning strategy for the Duplicator in the k-round Ehrenfeucht-Fraïssé game on structures $\mathfrak{M}$ and $\mathfrak{N}$. By Lemmas 4.10 and 4.11, the Duplicator can guarantee that the partial isomorphism defined after the kth round of play admits a k-good extension. ⊣

COROLLARY 4.13. $\mathfrak{M} \equiv_r \mathfrak{N}$.

PROOF. The empty partial isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$ has a 0-good extension by Lemma 4.10 (namely the identity function on the set $B \cup D \cup B^D \cup (B^D)^D$). Therefore, $\mathfrak{M}$ and $\mathfrak{N}$ are r-equivalent by Corollary 4.12. ⊣

Since $\mathfrak{M} \in \mathscr{K}_{\mathrm{even}}$ and $\mathfrak{N} \notin \mathscr{K}_{\mathrm{even}}$ and $r \in \mathbb{N}$ was taken to be arbitrary, Corollary 4.13 implies the main theorem of this section.

THEOREM 4.14. $\mathscr{K}_{\mathrm{even}}$ *is not first-order definable.*

PROOF. Toward a contradiction, assume there exists a first-order sentence Ψ defining $\mathscr{K}_{\mathrm{even}}$. Let r be the quantifier rank of Ψ, and let $\mathfrak{M}$ and $\mathfrak{N}$ be as constructed above with respect to this r. We now have:

$$\begin{array}{ll} \mathfrak{M} \models \Psi & \text{since } \mathfrak{M} \in \mathscr{K}_{\mathrm{even}}, \\ \mathfrak{M} \equiv_r \mathfrak{N} & \text{by Corollary 4.13}, \\ \mathfrak{N} \models \Psi & \text{by definition of } \equiv_r. \end{array}$$

But since $\mathfrak{N} \notin \mathscr{K}_{\text{even}}$, this contradicts our assumption that Φ defines $\mathscr{K}_{\text{even}}$. ⊣

As a corollary, we have proved our main theorem (Theorem 4.14) that successor-invariant logic $(\text{FO} + \text{succ})_{\text{inv}}$ is more expressive than FO.

§5. Conclusion. Theorem 4.14 can be somewhat strengthened.

COROLLARY 5.1. $(\text{FO}+\mathscr{C})_{\text{inv}}$ *is more expressive than* FO *where* $\mathscr{C}$ *is the class of permutations on finite sets with at most* k *cycles, for any fixed positive integer* k.

PROOF. We claim that $\mathscr{K}_{\text{even}}$ can be defined in the logic $(\text{FO} + \mathscr{C})_{\text{inv}}$, thus witnessing the separation $\text{FO} \subsetneqq (\text{FO} + \mathscr{C})_{\text{inv}}$. The formula Φ (Definition 3.5) no longer defines $\mathscr{K}_{\text{even}}$. However, we can suitably adapt Φ after observing that for a suitable choice of parameters F, F', G, G', the formula $Ⓢ(a, a', F, F', G, G')$ defines a permutation of A with at most k cycles. Therefore, we modify Φ to express that there exists a function $A \longrightarrow \{0, 1\}$ such that

- $a \longmapsto 0 \Longrightarrow Ⓢ(a) \longmapsto 1$ for all $a \in A$, and
- $|\{a \in A : a \longmapsto 1 \text{ and } Ⓢ(a) \longmapsto 1\}|$ is an even number $\leq k$.

Since k is fixed, this modified Φ can be expressed by a single $\mathscr{C}$-invariant sentence. ⊣

As another corollary of Theorem 4.14, we obtain the non-inclusion of epsilon-invariant logic in successor-invariant logic.

COROLLARY 5.2. $(\text{FO} + \varepsilon)_{\text{inv}} \not\subseteq (\text{FO} + \text{succ})_{\text{inv}}$.

PROOF. Let $\mathscr{K}^P_{\text{even}}$ denote the relativization of class $\mathscr{K}_{\text{even}}$ by a new unary predicate P. That is, $\mathscr{K}^P_{\text{even}}$ is the class of $(\sigma \cup \{P\})$-structures $\mathfrak{M}$ whose relativized reducts $\mathfrak{M}|^P_\sigma$ are $\mathscr{K}_{\text{even}}$-structures. $\mathscr{K}^P_{\text{even}}$ is epsilon-invariantly definable in the precisely manner described in Remark 3.9.

However, $\mathscr{K}^P_{\text{even}}$ is not definable in successor-invariant logic. To see this, consider successor relations which put a large number of non-P elements between any two elements of P. If $\mathfrak{M}'$ and $\mathfrak{N}'$ are $(\sigma \cup \{P, S\})$-structures with such "spaced out" successor relations and relativized reducts $\mathfrak{M}'|^P_\sigma = \mathfrak{M}$ and $\mathfrak{N}'|^P_\sigma = \mathfrak{N}$ (for $\mathfrak{M}$ and $\mathfrak{N}$ as defined in Definition 4.6), then $\mathfrak{M}'$ and $\mathfrak{N}'$ are clearly also indistinguishable by sentences of quantifier rank r. Therefore, $\mathscr{K}^P_{\text{even}}$ is not definable in successor-invariant logic. ⊣

An interesting question raised by this work concerns derangement-invariant logic. A *derangement* of a set X is a permutation d of X having no fixed points (i.e., such that $d(x) \neq x$ for all $x \in X$). Let $\mathscr{D}$ be the class of finite derangements, that is, structures consisting of a set X with a single binary relation D which is the graph of a derangement on X.

QUESTION 5.3. Is $(\text{FO} + \mathscr{D})_{\text{inv}}$ more expressive than FO?

REMARK 5.4. It is a simple exercise modifying $\mathscr{K}_{\text{even}}$ to get a class which is $(\text{FO} + \text{succ})_{\text{inv}}$-definable but not $(\text{FO} + \mathscr{D})_{\text{inv}}$-definable, thus proving the separation $(\text{FO} + \mathscr{D})_{\text{inv}} \subsetneqq (\text{FO} + \text{succ})_{\text{inv}}$.

REMARK 5.5. If instead of $\mathscr{D}$ we consider the class $\mathscr{C}$ of all finite permutations, then $(\text{FO} + \mathscr{C})_{\text{inv}}$ is equivalent in expressive power to FO. This is clear since

the identity permutation is uniformly definable in all structures by the single formula $x = y$. In general, for a class $\mathscr{C}$ to satisfy $\mathrm{FO} \subsetneq (\mathrm{FO} + \mathscr{C})_{\mathrm{inv}}$, there can exist no such uniform interpretation of a subclass of $\mathscr{C}$-structures.

Another open question is the precise relation between epsilon-invariant logic and order-invariant logic.

QUESTION 5.6 ([3, 10]). Is $(\mathrm{FO} + \varepsilon)_{\mathrm{inv}}$ more expressive than $(\mathrm{FO} + \varepsilon)_{\mathrm{inv}}$?

Figure 2 depicts the known separations among first-order logic and a few of its $\mathscr{C}$-invariant extensions. Question marks indicate relationships that are not completely determined.

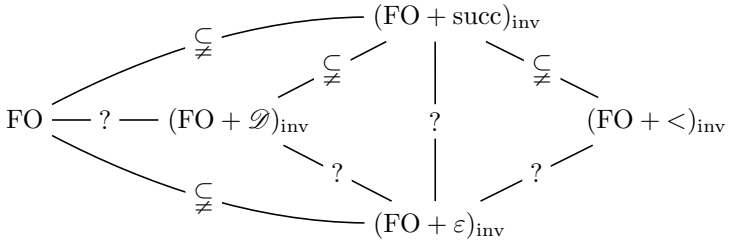

FIGURE 2. Known separations among first-order logic and a few of its $\mathscr{C}$-invariant extensions.

Finally, the reader is referred to the article [4] in which a few ideas developed here are used to construct elementary explicit (non-probabilistic) finite graphs and tournaments satisfying certain extension axioms.

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