BUNDLES OVER CONNECTED SUMS

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Abstract. A principal bundle over the connected sum of two manifolds need not be diffeomorphic or even homotopy equivalent to a non-trivial connected sum of manifolds. We show however that the homology of the total space of a bundle formed as a pullback of a bundle over one of the summands is the same as if it had that bundle as a summand. See Theorem 3.3. An application appears in [2].

Examples are given, including one where the total space of the pullback is not homotopy equivalent to a connected sum with that as a summand and some in which it is.

Finally, we describe the homology of the total space of a principal $U(1)$ bundle over a 6-manifold of the type described by Wall's theorem. It is a connected sum of an even number of copies of $S^3 \times S^4$ with a 7-manifold whose homology is $\mathbb{Z}/k$ in degree 4 (and $\mathbb{Z}$ in degrees 0 and 7, and zero in all other degrees).

1. Introduction

Let A be a connected sum $A \cong B\#C$ of n-manifolds. See for example Hatcher [1] for the definition of connected sum. Let $F \to L \to C$ be a bundle over C where F is a manifold.

Using the definition we get a map $A \to C$. Let $F \to M \to A$ be the pullback of the bundle $F \to L \to C$ to A.

Letting B' denote the complement of a chart in B and setting $X' := (B' \times F)/(* \times F)$ we prove the following. There is a cofibration $M \to L \to \Sigma X'$ for which the corresponding long exact homology/cohomology sequences split to give

$$H_*(M) \cong H_*(X') \oplus H_*(L) \qquad \text{and} \qquad H^*(M) \cong H^*(X') \oplus H^*(L).$$

(See Theorems 3.1 and 3.3.)

These results suggest the possibility that M is the connected sum of L and some manifold X whose $(n-1)$ skeleton is homotopy equivalent to X' but we give an example to show that this is not necessarily the case. (See Example 3.4). As we shall see, if $M \simeq X\#L$ then the cofibration sequence $X' \to M \to L$ would have to split to give $M \simeq X' \vee L$, but this fails in Example 3.4.

In the final section, we consider bundles over some 6-manifolds including the case where A is a symplectic manifold and the M is the total space of its associated prequantum line bundle. We find that in that case $M \simeq \#^{2r}(S^3 \times S^4)\#L$ where L is a 7-manifold whose nonzero cohomology groups are $\mathbb{Z}$ in degrees $0, 7$ and $\mathbb{Z}/k$ in degree 4, where r and k are determined by the cohomology of A. (See Theorem 4.1.)

The authors would like to thank Sebastian Chenery who pointed out an error in an earlier version of this paper.

For topological spaces X and Y, let $X \cong Y$ denote "X is homeomorphic to Y" and let $X \simeq Y$ denote "X is homotopy equivalent to Y".

2. Connected Sums

Let D^n denote the closed disk $D^n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$.

Lemma 2.1. *For any points a, b in the interior of D^n there exists a self-diffeomorphism $f : D^n \to D^n$ such that $f(a) = b$ and $f|_{\partial D^n}$ is the identity.*

Proof. Set $f(a) = b$. For $x \neq a$, let X_x be the point at which the production of the line segment joining a to x meets ∂D^n. Then

$$x = ta + (1-t)X_x$$

for some t. Set $f(x) = tb + (1-t)X_x$. □

More generally, we have

Lemma 2.2. *Let U_p, V_q be subcharts of D^n. Then there exists a self-diffeomorphism $f : D^n \to D^n$ such that the restriction of f to U_p is the standard diffeomorphism on open balls and such that $f|_{\partial D^n}$ is the identity.*

For a point p in an n-manifold X, define a subchart around p to be an open neighbourhood U_p of p which is diffeomorphic to an open ball in $\mathbb{R}^n$ within some chart of X.

For a connected n-manifold X, let $X' = X \setminus D^n$ denote the complement of a subchart of X.

Lemma 2.3. *Up to diffeomorphism, X' is independent of the choice of the subchart removed.*

Proof. Let U_p, U_q be subcharts of X. In the special case where there exists a chart W containing both $\bar{U}_p$ and $\bar{U}_q$ this follows from the earlier lemma. Then for arbitrary U_p, U_q, find a finite (by compactness) chain of charts connecting U_p to V_q, using connectivity.

□

After removal of the subchart there is a deformation retraction

$$X' \simeq X^{(n-1)}$$

to the $(n-1)$-skeleton of X. Let $f_X : S^{n-1} \to X'$ denote the attaching map of the top cell of X.

Suppose that X, Y are simply connected oriented n-manifolds.

In a connected sum, $X\#Y = X' \cup_{S^{n-1}\times I} Y'$ (where the orientation on one of the inclusions $S^{n-1} \times \{0\} \hookrightarrow X'$ or $S^{n-1} \times \{1\}$ is reversed so that $X\#Y$ inherits an orientation), there is a canonical projection $X\#Y \to X$. Similarly we have $X\#Y \to Y$. The canonical projections $X\#Y \to X$ and $X\#Y \to Y$ preserve the orientation class. That is, they induce isomorphisms on $H_n(\ )$.

Collapsing the centre of the tube $S^{n-1} \times I$ within $X\#Y$ gives a map $X\#Y \to X' \vee Y'$. If we form $(X\#Y)'$ by choosing the subchart to be removed to be within the centre of the tube then collapsing to produce $X' \vee Y'$ has collapsed a contractible subset of $(X\#Y)'$ giving a homotopy equivalence $(X\#Y)' \xrightarrow{\simeq} X' \vee Y'$.

By writing $S^n = S^n\#S^n$ and considering naturality of the pinch we see that the homotopy class of the attaching map of the top cell in $X\#Y$ is given by $f_{X\#Y} = f_X + f_Y$ within $\pi_n(X) \oplus \pi_n(Y) \subset \pi_n(X' \vee Y')$.

Choosing the subchart to be removed from $X\#Y$ to be within Y' gives a (non-canonical) inclusion $X' \hookrightarrow (X\#Y)'$ with $(X\#Y)'/X' \cong Y'$. The composite $X' \hookrightarrow (X\#Y)' \to X$ with the canonical projection is an injective map from a compact Hausdorff space, so it is a homeomorphism to its image. Composing with the inverse of this homeomorphism is a left splitting of the inclusion $X' \hookrightarrow (X\#Y)'$. Similarly there is a left splitting of the inclusion $Y' \hookrightarrow (X\#Y)'$.

Lemma 2.4. *Let M be a closed n-manifold and let $A \subset M'$ be a closed n-dim subset of M with $\partial\bar{A} \cong S^{n-1}$. Then $M \cong N\#X$ for some manifolds N and X with $N' = A$. Furthermore the canonical projection $M' \to N' = A$ is a left splitting of the inclusion $A \hookrightarrow M'$.*

Proof. Set $\hat{X} := M \setminus A$. Then $\hat{X}$ is a manifold-with-boundary with $\partial\hat{X} = \partial\bar{A}$. Let $T \cong S^{n-1} \times I$ be a tubular neighbourhood of $\partial\hat{X}$ in $\hat{X}$ and set $X' := \hat{X} \setminus T$. Then

$$M = \bar{A} \cup_{S^{n-1}\times\{0\}} T \cup_{S^{n-1}\times\{1\}} \overline{X'}$$

so $M = N\#X$ where $N = A \cup_{(T\times\{0\})} D^n$ and $X = X' \cup_{(T\times\{1\})} D^n$. By construction $A \hookrightarrow M' \to N' = A$ is the identity on A. $\square$

3. The cofibration sequence associated to a bundle over a connected sum

For definitions and properties of principal cofibrations used in this section see pp. 56–61 of [3].

Let B, C be closed orientable n-manifolds and let $A := B\#C$. Suppose

$$F \to L \to C$$

is a (locally trivial) fibre bundle whose fibre F is an orientable manifold. Then L is a manifold of dimension $n + \dim(F)$, which we will denote by m.

Let $F \to M \to A$ be the pullback of the bundle under the canonical projection $A \to C$. The total space M is a manifold of dimension m.

Let $\hat{L}$ be the total space of the restriction of the bundle to

$$C' := C \setminus \text{chart}.$$

By definition,

$$A = B' \cup_{S^{n-1}\times I} C'$$

where by construction, the restriction of the bundle to B' is trivial.

Taking inverse images under the bundle projection $M \twoheadrightarrow A$ gives

$$M = (B' \times F) \cup_{(S^{n-1}\times I\times F)} \hat{L}.$$

In other words, we have

$$\begin{tikzcd}
S^{n-1} \times I \times F \arrow[r, hook] \arrow[d] & (B' \times F) \arrow[r] \arrow[d] & (B' \times F)/(S^{n-1} \times I \times F) \arrow[d, equal] \\
\hat{L} \arrow[r, hook] & M \arrow[r] & M/\hat{L}
\end{tikzcd}$$

where the left square is a pushout.

The space

$$\begin{aligned}
M/\hat{L} &= (B' \times F)/(S^{n-1} \times I \times F) \\
&= \big(B'/(S^{n-1} \times I) \times F\big)/(* \times F) \\
&= (B \times F)/(* \times F).
\end{aligned}$$

has the same homology as $B \vee (B \wedge F)$. In fact, if F is a suspension then $(B \times F)/(* \times F) \simeq B \vee (B \wedge F)$. (Selick, [3] Prop 7.7.8)

Set $X' := (B' \times F)/(* \times F)$.

Theorem 3.1. *There is a cofibration diagram*

$$\begin{tikzcd}
M' \arrow[r] \arrow[d] & L' \arrow[r] \arrow[d] & \Sigma X' \arrow[d, equal] \\
M \arrow[r] & L \arrow[r] & \Sigma X'
\end{tikzcd}$$

(i.e. the rows are cofibrations and the right square is a pushout.)

Proof. We had

$$M/\hat{L} = (B' \times F)/(S^{n-1} \times I \times F)$$

Also, since $L = \hat{L} \cup_{S^{n-1} \times I \times F} F$ we have

$$L/\hat{L} = (S^n \times F)/(* \times F)$$

(which can be regarded as the special case $B = S^n$ of the preceding). Thus we have a diagram

$$\begin{tikzcd}
 & & X' \arrow[d] \\
\hat{L} \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & M/\hat{L} = (B \times F)/(* \times F) \arrow[d] \\
\hat{L} \arrow[r] & L \arrow[r] \arrow[d] & L/\hat{L} = (S^n \times F)/(* \times F) \arrow[d] \\
 & \Sigma X' \arrow[r, equal] & \Sigma X'
\end{tikzcd}$$

in which the top right square is a pushout, the rows and right columns are cofibrations and which yields the cofibration $M \to L \to \Sigma X'$. Deleting a chart from L and deleting its preimage from M gives the first row of the theorem. □

From the long exact homology sequence of the cofibration we get

Corollary 3.2. *The lift $M \to L$ of the canonical projection preserves the orientation class. That is, it induces isomorphisms on $H_m(\ )$, where $m = \dim L = \dim M$.*

This Corollary can be proved in other ways such as naturality of the Serre spectral sequence.

Let $f : X \to Y$ be a differentiable map between compact oriented m-manifolds. Let $D_X : H^k(X) \cong H_{n-k}(X)$ and $D_Y : H^k(Y) \cong H_{n-k}(Y)$ be the Poincaré Duality isomorphisms. Suppose f has degree λ (multiplies by λ on $H_n(\ )$). Then $f_* \circ D_X \circ f^* = \lambda D_Y$. In particular, if f preserves the orientation class (that is, has degree 1) then f^* is injective and f_* is surjective. Applying this to $M \to L$ shows

Theorem 3.3. *(Decomposition Theorem)*

In the long exact homology sequence of the cofibration, the connecting map $\partial : H_q(L) \to H_{q-1}(X')$ is zero. Likewise, in the long exact cohomology sequence, the map $\delta : H^{q-1}(X') \to H^q(L)$ is zero. Thus if $H_*(X')$ is torsion free then for $0 < q < m$ we have

$$H_q(M) \cong H_q(X') \oplus H_q(L) \qquad \textit{and} \qquad H^q(M) \cong H^q(X') \oplus H^q(L).$$

This suggests that perhaps there is a manifold X such that $M \simeq X\#L$ where X is homotopy equivalent to the one-point compactification of X', but this is not necessarily true.

Example 3.4. Consider $A = \mathbb{C}P^2$ and write $A = B\#C$ where

$$B = \mathbb{C}P^2 \qquad \text{and} \qquad C = S^4.$$

Consider the trivial bundle $S^7 \times S^4 \to S^4$. Then $M = S^7 \times \mathbb{C}P^2$; $B' = S^2$; $C' = *$; $A' = B' \vee C' = S^2$ while

$$\begin{aligned} M' &= (F \times A)' = (F \times A') \cup_{F' \times A'} (F' \times A) \\ &= (S^7 \times S^2) \cup_{* \times S^2} (* \times \mathbb{C}P^2) = \mathbb{C}P^2 \vee S^7 \vee S^9 \end{aligned}$$

and $L = S^4 \times S^7$ so $L' = S^4 \vee S^7$. Our cofibration is

$$(S^2 \times S^7)/(* \times S^7) \to M' \to S^4 \vee S^7$$

which becomes $S^2 \vee S^9 \to \mathbb{C}P^2 \vee S^7 \vee S^9 \to S^4 \vee S^7$. This does not split so in this example M does not become homotopy equivalent to $X\#L$ for any X.

4. Bundles over 6-manifolds

Let A be a simply connected 6-manifold such that $H^*(A)$ is torsion-free. Suppose $H^2(A) = \mathbb{Z}$.

Let $x \in H^2(A)$ be a generator and let $V \in H^6(A)$ be the volume form. Then $x^3 = kV$ for some integer k.

By Wall [4], we can write $A = B\#C$ where $B = (S^3 \times S^3)^{\#r}$ for some r and C is a simply connected torsion-free 6-manifold with $H^3(C) = 0$ and $H^2(C) = \mathbb{Z}$.

Associated to x there are complex line bundles over A and C classified by x. Let M and L denote the sphere bundles of these line bundles. Then there are S^1-bundles $S^1 \to M \to A$ and $S^1 \to L \to C$. Note that the long exact homotopy sequence tells us that $\pi_1(M) = \pi_2(M) = 0$ and $\pi_q(M) = \pi_q(A)$ for $q \neq 1$.

Although M is a S^1 bundle over A, it does not immediately follow from Wall's result that M also admits a decomposition as a connected sum. We shall see that this is in fact true. This is the content of our Theorem 4.1 below.

As in Ho-Jeffrey-Selick-Xia [2] we calculate that the cohomology of the 7-manifold L is given by $H^q(L) = \begin{cases} \mathbb{Z} & q = 0, 7; \\ \mathbb{Z}/k & q = 4; \\ 0 & \textit{otherwise}. \end{cases}$

Theorem 4.1. *We have*

$$M \simeq \#^{2r}(S^3 \times S^4)\#L,$$

where the homology of the space L is specified above.

Proof. In the notation of the preceding section applied to $S^1 \to M \to A$ we have $B' = \vee_{2r} S^3$, $L' = P^4(k)$ and

$$X' := (B' \times S^1)/(* \times S^1) \simeq B' \vee (B' \wedge S^1) \vee_{2r} (S^3 \vee \Sigma^3 S^1)$$

where $P^n(k)$ denotes the Moore space $S^{n-1} \cup_k e^n$. Thus our cofibration sequence becomes

$$\vee_{2r}(S^3 \vee \Sigma^3 S^1) \to M' \to P^4(k)$$

or equivalently

$$\vee_{2r}(S^3 \vee S^4) \to M' \to P^4(k).$$

The composition of the bundle map $M' \to A'$ with the canonical projection $A' \to B'$ provides a splitting of the restriction of

$$\vee_{2r}(S^3 \vee S^4) \to M'$$

to $\vee_{2r} S^3$.

For degree reasons, the cofibration

$$\vee_{2r}(S^3 \vee S^4) \to M' \to P^4(k)$$

is principal, induced from some attaching map $P^3(k) \to \vee_{2r}(S^3 \vee S^4)$ whose image (for degree reasons) lands in $\vee_{2r} S^3$. Since the restriction of $\vee_{2r}(S^3 \vee S^4) \to M'$ to $\vee_{2r} S^3$ splits, this implies that this attaching map is trivial. Thus the cofibration splits to give

$$M' \simeq \vee_{2r}(S^3 \vee S^4) \vee_{2r} P^4(k).$$

To obtain M from M' we attach the top cell giving

$$H^q(M) = H^q(M') \oplus H^q(S^7) = \begin{cases} \mathbb{Z} & q = 0, 7; \\ \mathbb{Z}^{2r} & q = 3; \\ \mathbb{Z}^{2r} \oplus \mathbb{Z}/k & q = 4; \\ 0 & \text{otherwise.} \end{cases}$$

Letting $\tilde{V}$ denote the generator of $H^7(M)$, using Poincaré duality we can pair the generators $\langle u_1, u_2, \ldots u_{2r}\rangle$ of $\mathbb{Z}$ in degrees 3 with the generators $\langle v_1, v_2, \ldots v_{2r}\rangle$ of $\mathbb{Z}$ in degrees 4 so that $u_i v_j = \delta_{ij}\tilde{V}$. If we reduce to $\mathbb{Z}/k$ coefficients, there is also a nonzero cup product ab where a, b are generators of $H^3(M; \mathbb{Z}/k)$ and $H^4(M; \mathbb{Z}/k)$ respectively.

Examining the cohomology of M, we see that

$$H^*(M) = H^*\big(\#^{2r}(S^3 \times S^4)\#L\big)$$

where $H^q(L) = \begin{cases} \mathbb{Z} & q = 0, 7 \\ \mathbb{Z}/k & q = 4 \\ 0 & \text{otherwise.} \end{cases}$

The attaching maps f_M and $f_{\#^{2r}(S^3\times S^4)\#L}$ are both

$$[\iota_1^3, \iota_1^4] + [\iota_2^3, \iota_2^4] + \ldots + [\iota_r^3, \iota_r^4] + f_L$$

where the Whitehead product $[\iota^3, \iota^4]$ is the attaching map

$$f_{S^3\times S^4},$$

and so

$$M \simeq \#^{2r}(S^3 \times S^4)\#L.$$

$\square$

References

[1] A. Hatcher, *Algebraic Topology.* Cambridge University Press, 2002.

[2] N. Ho, L. Jeffrey, P. Selick, E. Xia, Flat connections and the commutator map for $SU(2)$. *Oxford Quart. J. Math.,* to appear (in volume in memory of Michael Atiyah)

[3] P. Selick, *Introduction to Homotopy Theory.* AMS (Fields Institute Monographs vol. 9), 1997.

[4] C.T.C. Wall, Classification problems in differential topology. V. On certain 6-manifolds. *Invent. Math.* **1** (1966), 355–374; corrigendum, ibid., **2** (1966), 306.

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