A Minimal Subsystem of the Kari-Culik Tilings

Jason Siefken*

Department of Mathematics and Statistics

University of Victoria
P. O. Box 3045
Victoria, BC
Canada V8P 5C2
siefkenj@uvic.ca

September 28, 2015

Abstract

The Kari-Culik tilings are formed from a set of 13 Wang tiles that tile the plane only aperiodically. They are the smallest known set of Wang tiles to do so and are not as well understood as other examples of aperiodic Wang tiles. We show that the $\mathbb{Z}^2$ action by translation on a certain subset of the Kari-Culik tilings, namely those whose rows can be interpreted as Sturmian sequences (rotation sequences), is minimal. We give a characterization of this space as a skew product as well as explicit bounds on the waiting time between occurrences of $m \times n$ configurations.

1 Introduction

The Kari-Culik tilings are tilings of the plane by the 13 square tiles specified in Figure 1. Translations, but not rotations, of tiles are allowed, and two tiles may share an edge if the labels of those edges match.

Published in 1995, they are the smallest known set of square tiles admitting only aperiodic tilings of the plane. Further, unlike the Robinson tilings and other well-known aperiodic tilings, the current proofs of aperiodicity for the Kari-Culik tilings are based on number-theoretic arguments with no known hierarchical explanation.

Eigen, Navarro, and Prasad provide a detailed exposition and proof of the aperiodicity and existence of Kari-Culik tilings [3]. In their proof, Eigen et al. show the existence of

*Supported by the University of Victoria and NSERC

2000 Mathematics Subject Classification. Primary: 52C20, 05B45, 5223. Secondary: 37E05.

Key words: aperiodic tiling, Wang tiles, aperiodic SFT, Kari-Culik tiling.

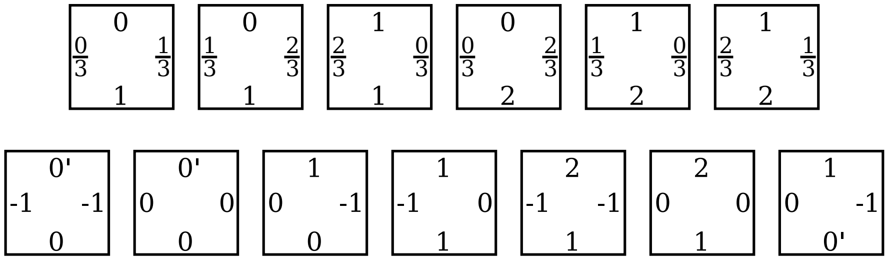


Figure 1: List of the 13 Kari-Culik tiles.

Kari-Culik tilings arising from Sturmian sequences (defined in the next section). It was unknown if there were Kari-Culik configurations not arising from Sturmian sequences until Durand, Gamard, and Grandjean showed that the set of all Kari-Culik tilings has positive topological entropy [2]. Since the number of Sturmian sequences of length n is bounded by a polynomial $p(n)$ [1], the fact that $n \log p(n)/n^2 \to 0$ shows that the subset of tilings arising from Sturmian sequences must have zero-entropy. This implies the existence of non-Sturmian Kari-Culik tilings.

In this paper, however, we will only concern ourselves with a subset of the Kari-Culik tilings. Let KC be the subset of the Kari-Culik tilings whose rows form (generalized) Sturmian sequences.

Theorem A. *The $\mathbb{Z}^2$ action by translation on KC is conjugate to a skew product acting on the space $[1/3, 2] \times \varprojlim \mathbb{R}/(6^n\mathbb{Z})$.*

Theorem B. *The $\mathbb{Z}^2$ action by translation on KC is minimal.*

Theorem C. *If A is an $m \times n$ sub-configuration of some point in KC, every sub-configuration of size at least $6^{34.464m+25}n^{34.464} \times 6^{5m+3}n^4$ of a point in KC contains a copy of A.*

Theorem C gives explicit bounds on the maximum waiting time for any $m \times n$ configuration. In particular these bounds are of the form $\exp(m) \cdot \text{polynomial}(n)$.

2 Definitions

A Wang tiling is a class of nearest-neighbor subshift of finite type (SFT) on $\mathbb{Z}^2$ whose rules can be given by square tiles with labeled edges. We will interpret valid Kari-Culik tilings as a nearest-neighbor SFT on 13 symbols with the adjacency rules given by Figure 1. The set of the 13 Kari-Culik tiles will be called $\mathfrak{K}$.

T, S are the usual horizontal and vertical shifts on $\mathbb{Z}^2$. We say a configuration $x \in \mathfrak{K}^{\mathbb{Z}^2}$ is *aperiodic* if $T^a S^b x = x$ implies that $(a, b) = (0, 0)$. We denote by $\mathcal{O}(x) = \{T^a S^b x : a, b \in \mathbb{Z}\}$ the orbit of x and $\mathcal{O}_X(x) = \{X^a x : a \in \mathbb{Z}\}$ the orbit of x under the map X where $X \in \{T, S\}$.

Given a vector $\vec{x} \in \mathbb{R}^{\mathbb{Z}}$, x_i is the ith component of $\vec{x}$. If $x = (\ldots, x_{-1}, x_0, x_1, \ldots) \in \mathfrak{K}^{\mathbb{Z}}$, then $(x)_i^j = (x_i, \ldots, x_j)$ is the subword of x from position i to j and if $x \in \mathfrak{K}^{\mathbb{Z}^2}$, $(x)_i$ refers to the ith row of x. Further, if $A \subset \mathbb{Z}^2$ and $x \in \mathfrak{K}^{\mathbb{Z}^2}$, then $x|_A$ is the restriction of x to the indices in A. We denote by d the usual metric on bi-infinite sequences. That is,

$$d(x, y) = \inf\{2^{-i} : (x)_{-i}^i = (y)_{-i}^i\}.$$

Definition (a-valuation)**.** *For $q \in \mathbb{Q}$, we denote by $|q|_a$ the a-valuation of q. That is, if*

$$q = \prod_{p \text{ prime}} p^{n_p}$$

is the prime decomposition of q with $n_p \in \mathbb{Z}$, then $|q|_a = -n_a$.

Definition (Rotation Sequence)**.** *A* rotation sequence *corresponding to the parameters $\alpha, t \in \mathbb{R}$ is the sequence $x = \mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ or $x' = \mathbf{R}_{\lceil \cdot \rceil}(\alpha, t)$ where*

$$(x)_i = \lfloor i\alpha + t \rfloor - \lfloor (i-1)\alpha + t \rfloor$$

and

$$(x')_i = \lceil i\alpha + t \rceil - \lceil (i-1)\alpha + t \rceil .$$

The parameters α, t are called the angle *and the* phase *of the sequence.*

Definition (Sturmian Sequence)**.** *A sequence x is a* Sturmian sequence *if $x = \mathbf{R}_{\lfloor \cdot \rfloor}(\alpha, t)$ or $x = \mathbf{R}_{\lceil \cdot \rceil}(\alpha, t)$ for some $\alpha, t \in \mathbb{R}$. We call α the* angle *of x and t a* phase *of x. $\mathcal{S}$ denotes the set of all Sturmian sequences and $\bar{\mathcal{S}}$ denotes its closure under d. $\bar{\mathcal{S}}$ is called the set of* generalized Sturmian sequences*.*

Unlike some authors, we allow Sturmian sequences to be periodic. A consequence is that although the angle of any Sturmian sequence is uniquely defined as the average of its symbols, if the angle of a Sturmian sequence is rational (and hence the sequence is periodic), its phase is not uniquely defined. Further, we allow Sturmian sequences to consist of symbols other than $\{0, 1\}$. For a detailed exposition of Sturmian sequences and their equivalent characterizations, see [4, 6]. It is also worth noting that the set of generalized Sturmian sequences is strictly larger than the set of Sturmian sequences. For example $x = (\ldots, 0, 0, 1, 0, 0, \ldots) \in \bar{\mathcal{S}}$ but $x \notin \mathcal{S}$.

Definition (Balanced Sequence)**.** *A sequence x is a* balanced sequence *if for any two subwords $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ of x of equal length, we have $|\sum u_i - \sum v_i| \leq 1$.*

Every Sturmian sequence is a balanced sequence [4].

Proposition 1. *Every generalized Sturmian sequence is a balanced sequence.*

Proof. Let $x \in \bar{\mathcal{S}}$ and suppose there exist $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ such that $|\sum u_i - \sum v_i| > 1$. Since x is a limit point of $\mathcal{S}$, there must exist $x' \in \mathcal{S}$ such that u, v are both subwords of x'. However, x' is a Sturmian sequence and therefore balanced, which is a contradiction. ∎

Definition. *For a sequence x, define $\alpha(x) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{|i|\leq n} x_i$ to be the average of its symbols if it exists.*

For any $x \in \bar{\mathcal{S}}$, $\alpha(x)$ is defined. This follows from the fact that $x \in \bar{\mathcal{S}}$ is a balanced sequence and therefore the average of its symbols exist. Now, from the definition of a rotation sequence, we see if $x \in \mathcal{S}$, then $\alpha(x)$ is the angle of x.

Definition. *For a Sturmian sequence x, define $t(x) = \inf\{t : t$ is a phase for $x\}$.*

Proposition 2. *For $x \in \bar{\mathcal{S}}$, if $\alpha(x) \notin \mathbb{Q}$ then $x \in \mathcal{S}$ and $x = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha(x), t(x))$ or $x = \mathbf{R}_{\lceil\cdot\rceil}(\alpha(x), t(x))$.*

Proof. Since $\alpha(x) \notin \mathbb{Q}$, we necessarily have that x is not eventually periodic. Since x is also a balanced sequence, x is an aperiodic Sturmian sequence [4]. The proposition now follows from the fact that an irrational Sturmian sequence has a unique phase. ■

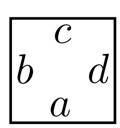
Definition. $\Phi : \mathfrak{K}^{\mathbb{Z}^2} \to \{0,1,2\}^{\mathbb{Z}^2}$ *is projection onto the bottom labels of tiles in $\mathfrak{K}$ followed by mapping the symbol $0'$ to 0.*

Definition. *The set $KC = \{x : x$ is a Kari-Culik tiling and $\Phi(x)$ consists of generalized Sturmian rows$\}$. $KC_{\mathbb{Q}^c} = \{x \in KC : (\Phi(x))_i$ has an irrational angle for all $i\}$.*

Extending notation, for $x \in KC$, we define $\alpha(x) = (\ldots, \alpha((\Phi(x))_0), \alpha((\Phi(x))_1), \ldots)$ and $t(x) = (\ldots, t((\Phi(x))_0), t((\Phi(x))_1), \ldots)$ to be the *angle vector* and *phase vector* of x (if they exist).

3 Kari-Culik Properties

Notice that for any Kari-Culik tiling, the rows fall into two distinct categories: those where every tile has left-right edge labels in $\{\frac{0}{3}, \frac{1}{3}, \frac{2}{3}\}$ and those where every tile has left-right edge labels in $\{0, -1\}$. We will call these rows as well as the tiles in each row *type* $\frac{1}{3}$ and *type* 2 respectively. The convention in this paper will be to refer to the labels of a tile in $\mathfrak{K}$ in clockwise order starting with the bottom label. That is, the labels a, b, c, d of a tile will correspond to the figure:



Part of the cleverness of the Kari-Culik tilings is that every tile satisfies the following.

Definition (Multiplier Property)**.** *A Kari-Culik tile with bottom, left, top, and right labels of a, b, c, d satisfies the relationship*

$$\lambda a + b = c + d \tag{1}$$

where $\lambda \in \{\frac{1}{3}, 2\}$ corresponds to the type of the tile. We also refer to λ as the multiplier *of the tile.*

Proposition 3. *Fix a Kari-Culik configuration x and let $r_0 = \Phi((x)_0)$ and $r_1 = \Phi((x)_1)$. Then, if the average of r_0 exists, it satisfies the relation*

$$\lambda\alpha(r_0) = \alpha(r_1)$$

where $\lambda \in \{\frac{1}{3}, 2\}$ is the type of $(x)_0$.

Proof. This is a direct result of the telescoping nature of the multiplier property when rewritten as $\lambda a - c = d - b$. Notice that in any row, every tile is the same type and therefore has the same multiplier. Let a_i be the bottom labels and c_i be the top labels of $(x)_0$. Summing along a central segment of length $2n+1$, we have

$$\lambda \sum_{i=-n}^{n} a_i - \sum_{i=-n}^{n} c_i = d - b \tag{2}$$

where b, d are the left and right labels of the central segment. Since

$$\alpha(r_0) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} a_i \qquad \text{and} \qquad \alpha(r_1) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=-n}^{n} c_i$$

and b, d are bounded, dividing both sides of Equation (2) by $2n+1$ and taking a limit produces the desired relationship. ■

It was further shown by Durand et al. that in fact the average of the bottom labels of any row in a Kari-Culik tiling exists, making the assumption that the average exists in Proposition 3 unnecessary [2].

Proposition 4. *For a Kari-Culik tiling x, $\alpha((\Phi(x))_i) \in [1/3, 2]$.*

Proof. As noted earlier, $\alpha((\Phi(x))_i)$ always exists. Fix i and let $\alpha = \alpha((\Phi(x))_i)$. Inspecting the tile set, we see that the largest symbol on the bottom of any tile is 2 and so $\alpha \leq 2$. To see that $\alpha \geq 1/3$, we will consider rows by type. For a row of type $\frac{1}{3}$, the smallest symbol appearing on the bottom is 1, and so $\alpha \geq 1 \geq 1/3$.

For a row of type 2, notice that the bottom labels may contain 0 or $0'$ but not both (if a row of type 2 had both 0 and $0'$ on the bottom, the row below it would need to have tiles of both type $\frac{1}{3}$ and type 2). If the bottom labels only contain 0, then the row below $(x)_i$ must be of type $\frac{1}{3}$. Inspecting the type $\frac{1}{3}$ tiles, we see that no more than two consecutive 0 symbols may occur as top labels and so $(x)_i$ cannot have more than two 0 symbols in a row as bottom labels giving $\alpha \geq 1/3$. Finally, notice that as bottom labels, all occurrences of $0'$ are isolated. Thus, if the row below $(x)_i$ is of type 2, $\alpha \geq 1/2 \geq 1/3$. ■

Definition. *For $x \in [1/3, 2]$, define*

$$\lambda_x = \begin{cases} 2 & \text{if } x \in [1/3, 1) \\ 1/3 & \text{if } x \in [1, 2] \end{cases} \qquad \text{and} \qquad f(x) = \lambda_x x = \begin{cases} 2x & \text{if } x \in [1/3, 1) \\ x/3 & \text{if } x \in [1, 2] \end{cases}.$$

Corollary 5. *Fix a Kari-Culik configuration x and let $r_i = \Phi((x)_i)$. Then,*

$$\alpha(r_{i+1}) = f(\alpha(r_i))$$

provided $\alpha(r_i) \neq 1$.

Proof. Since $\alpha(r_{i+1}) = \lambda\alpha(r_i)$ for some $\lambda \in \{\frac{1}{3}, 2\}$, the constraint that both $\alpha(r_{i+1}), \alpha(r_i) \in [1/3, 2]$ uniquely determines λ when $\alpha(r_i) \neq 1$. ∎

Even if $\alpha(r_i) = 1$, there are still only two options for $\alpha(r_{i+1})$ and as will be shown in Proposition 7, orbits under f are aperiodic, ensuring this can occur at most once.

Using observations about the multiplier of tiles and the averages of sequences of bottom labels, we can refine our classification of rows of the Kari-Culik tilings.

Definition. *Let $x \in KC$ and $r_i = (x)_i$ be the ith row of x. We define the* general type *of r_i based on the tiles in r_{i-1}, r_i, r_{i+1} in the following way.*

Type $\frac{1}{3}$*: r_i is of general type $\frac{1}{3}$ if r_i consists of type $\frac{1}{3}$ tiles.*

Type 2.1*: r_i is of general type 2.1 if r_i consists of type 2 tiles and r_{i+1}, r_{i-1} both consist of type $\frac{1}{3}$ tiles.*

Type 2.2t*: r_i is of general type 2.2t if r_i consists of type 2 tiles and r_{i+1} consists of type $\frac{1}{3}$ tiles while r_{i-1} consists of type 2 tiles.*

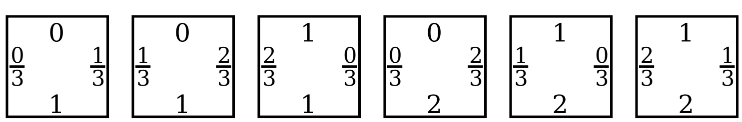
Type 2.2b*: r_i is of general type 2.2b if r_i consists of type 2 tiles and r_{i-1} consists of type $\frac{1}{3}$ tiles while r_{i+1} consists of type 2 tiles.*

We consider a pair of rows whose top row is of general type 2.2t and whose bottom row is of general type 2.2b as type 2.2.

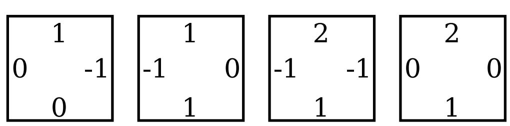
Since general type $\frac{1}{3}$ exactly corresponds to type $\frac{1}{3}$ and we have no previous definition for type 2.1, 2.2t, 2.2b, or 2.2, without ambiguity we may from now on refer to the general type of a row as simply the type of that row.

Proposition 6. *Let $x \in KC$ and $r_i = (x)_i$ be the ith row of x. The general type of r_i is unique and the tiles that may appear in r_i are contained in exactly one of the following (non-disjoint) sets based on general type.*

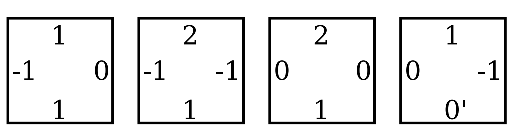
Type $\frac{1}{3}$:



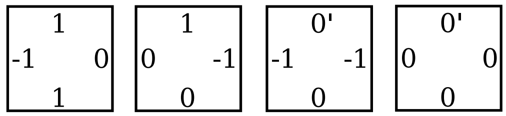
Type 2.1:



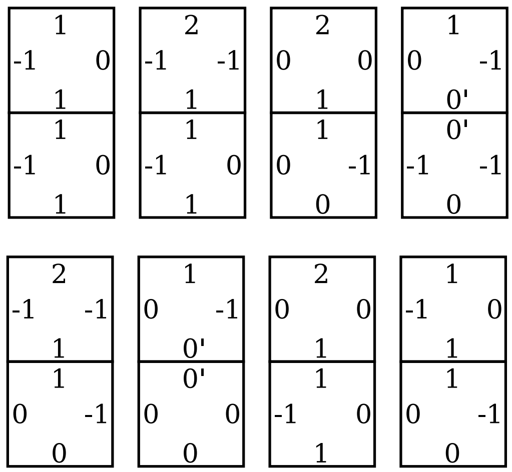
Type 2.2t:



Type 2.2*b:*



A pair of rows whose top tiles are type 2.2*t and bottom tiles are type* 2.2*b taken together and considered as type* 2.2 *consists of the stacked tiles:*



Proof. Fix a Kari-Culik tiling x. Let $r_i = (x)_i$ be the ith row of x and let λ_i be its multiplier. Let α be the average of the bottom labels of r_i. We will show something slightly stronger than is asked, namely that except for $\alpha \in \{1/2, 2/3, 1\}$, α uniquely determines the type r_i.

If $\alpha \in (1, 2]$, then $\lambda = \frac{1}{3}$, and so r_i must consist of tiles of type $\frac{1}{3}$, making r_i of general type $\frac{1}{3}$.

If $\alpha \in (1/2, 2/3)$, r_{i+1}, r_{i-1} must be of type $\frac{1}{3}$, and so $0'$ cannot occur as a label, making r_i of general type 2.1 and leaving the only available tiles those listed as type 2.1.

If $\alpha \in [1/3, 1/2)$, the rows r_i and r_{i+1} are both of type 2 and r_{i-1} and r_{i+2} are of type $\frac{1}{3}$. Thus, r_i must be of general type 2.2b and must consist of the tiles listed as type 2.2b.

Finally, if $\alpha \in (2/3, 1)$, r_{i-1} is of type 2 and r_{i+1} and r_{i-2} are both of type $\frac{1}{3}$. Thus, r_i must be of general type 2.2t and consists of the tiles listed as type 2.2t.

The tiles listed as type 2.2 consist of the ways to stack type 2 tiles to be compatible on tops and bottoms with type $\frac{1}{3}$ tiles and so correspond exactly to the cases where 2.2t and 2.2b tiles arise in consecutive rows.

In the remaining cases of $\alpha \in \{1/2, 2/3, 1\}$, the type of r_i is not strictly determined by α, but nonetheless, the tiles in r_i fall into one of the four categories and the classification is unique. ■

The tiles listed as type 2.1 have non-trivial intersection with the tiles listed as type 2.2t and type 2.2b, however since no two rows of type 2.1 occur consecutively, the categorization is unique. We call the pairs of tiles listed as type 2.2 *stacked tiles*. When we think of a row of a Kari-Culik tiling as being type 2.2, we may think of its multiplier as being 4 (since it is composed of two consecutive rows with multiplier 2).

Proposition 7 (Liousse [5]). *The map f is conjugate to an irrational rotation by* $\log 2/\log 6$.

Proof. An explicit conjugacy $\phi : [1/3, 2] \to [0, 1]$ is given by $\phi(x) = \frac{\log x + \log 3}{\log 6}$. ■

Analogously to the way Eigen et al. show the existence of Kari-Culik tilings through their Basic Tile Construction, we will show how to take an angle vector and a phase vector and produce a valid element of KC.

Definition (BC Property). *A pair of vectors* $(\vec{\alpha}, \vec{t}) \in [1/3, 2]^{\mathbb{Z}} \times [0, 1]^{\mathbb{Z}}$ *satisfies the* BC property *(Basic Construction property) if*

$$\lambda_i = \frac{\alpha_i}{\alpha_{i+1}} \in \{\tfrac{1}{3}, 2\}$$

and

$$\begin{array}{ll} 2t_i = t_{i+1} \bmod 1 & \text{if } \lambda_i = 2 \\ t_i = 3t_{i+1} \bmod 1 & \text{if } \lambda_i = \frac{1}{3} \end{array}$$

for all i.

Given a pair of vectors $(\vec{\alpha}, \vec{t})$ satisfying the BC property, we can construct a point $y \in KC$ via the following procedure. The tile at position m, n in y has bottom, left, top, and right edges given by

$$\begin{array}{rcl} \text{bottom} & = & \lfloor n\alpha_m + t_m \rfloor - \lfloor (n-1)\alpha_m + t_m \rfloor \\ \text{left} & = & \lambda \lfloor (n-1)\alpha_m + t_m \rfloor - \lfloor \lambda(n-1)\alpha_m + t_m \rfloor \\ \text{top} & = & \lfloor n\alpha_{m+1} + t_{m+1} \rfloor - \lfloor (n-1)\alpha_{m+1} + t_{m+1} \rfloor \\ \text{right} & = & \lambda \lfloor n\alpha_m + t_m \rfloor - \lfloor \lambda n\alpha_m + t_m \rfloor \end{array}$$

where $\lambda = \alpha_m/\alpha_{m+1}$. Further, if either the bottom or the top label is computed to be 0, then 0 is replaced with $0'$ if $\alpha_{m-1} \in [1/3, 1/2]$ (respectively $\alpha_m \in [1/3, 1/2]$). We can also do the same construction using $\lceil \cdot \rceil$ instead of $\lfloor \cdot \rfloor$. We call a tiling constructed in this way a *Basic Construction* with parameters $(\vec{\alpha}, \vec{t})$.

Proposition 8 (Robinson [8]). *If $(\vec{\alpha}, \vec{t})$ satisfies the BC property, then the resulting Basic Construction using either $\lfloor \cdot \rfloor$ or $\lceil \cdot \rceil$ is an element of KC.*

Proof. First observe that if y is the result of a Basic Construction, then $\Phi(y)$ consists of rows that are rotation sequences and therefore Sturmian. Further, by definition, the top labels of each row of y are guaranteed to be compatible with the bottom labels of the next row, and the right labels of each column of y are guaranteed to be compatible with the left labels of the next column.

The remainder of the proof involves checking for all ranges of α_m, t_m that the resulting bottom, left, top, and right labels correspond to an actual tile in $\mathfrak{K}$. The details of this are straightforward, and after substituting $t_{m+1} = 2t_m \bmod 1$ or $t_m = 3t_{m+1} \bmod 1$ depending on the ratio α_m/α_{m+1}, it requires only examining what cases result from the choice of α_m,t_m or α_m, t_{m+1}. ■

Proposition 9 provides a partial converse to Proposition 8. Proving that a tiling in $KC_{\mathbb{Q}^c}$, like tilings arising from a Basic Construction, can be expressed with one of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$, but never requires a mixture of both, is the bulk of the proof of Proposition 9.

Proposition 9. *If $y \in KC_{\mathbb{Q}^c}$ and $(\vec{\alpha}, \vec{t}) = (\alpha(\Phi(y)), t(\Phi(y)))$ are the angle and phase vectors of y, then y is the result of a Basic Construction arising from $(\vec{\alpha}, \vec{t})$ using either $\lfloor\cdot\rfloor$ or $\lceil\cdot\rceil$.*

Proof. First note that since $y \in KC_{\mathbb{Q}^c}$, the rows of $\Phi(y)$ are Sturmian sequences and therefore rotation sequences (since rows in $\bar{\mathcal{S}}\backslash\mathcal{S}$ are excluded).

We will first show that $(\vec{\alpha}, \vec{t})$ satisfies the BC property. Fix $k \in \mathbb{Z}$. Since Corollary 5 already shows that $\vec{\alpha}$ is determined by f and α_k (that is $\alpha_{k+i} = f^i(\alpha_k)$), we only need to show that either $t_{k+1} = 2t_k \bmod 1$ or $t_k = 3t_{k+1} \bmod 1$ in accordance with α_k. For simplicity, call $\alpha = \alpha_k$, $t = t_k$, and $t' = t_{k+1}$. Let $\lambda = \alpha_k/\alpha_{k+1}$ be the type of the kth row of y and let a_i, b_i, c_i, d_i be the bottom, left, top, and right labels of the ith tile in $(y)_k$. We divide the proof into two similar cases depending on λ.

Case $\lambda = 1/3$: We will assume the Sturmian sequences $(\Phi(y))_k$ and $(\Phi(y))_{k+1}$ may both be represented using $\mathbf{R}_{\lfloor\cdot\rfloor}$, but note that for every combination $(\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lfloor\cdot\rfloor})$, $(\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil})$, $(\mathbf{R}_{\lceil\cdot\rceil}, \mathbf{R}_{\lfloor\cdot\rfloor})$, and $(\mathbf{R}_{\lceil\cdot\rceil}, \mathbf{R}_{\lceil\cdot\rceil})$ of ways to represent $(\Phi(y))_k$ and $(\Phi(y))_{k+1}$, upon replacing $\lfloor\cdot\rfloor$ with $\lceil\cdot\rceil$ where appropriate, the same argument still works. By Corollary 5, $\alpha_{k+1} = \lambda\alpha_k = \lambda\alpha$, and so we have the following relationship for the bottom and top labels:

$$a_i = \lfloor i\alpha + t\rfloor - \lfloor (i-1)\alpha + t\rfloor \qquad \text{and} \qquad c_i = \left\lfloor \frac{i\alpha + 3t'}{3} \right\rfloor - \left\lfloor \frac{(i-1)\alpha + 3t'}{3} \right\rfloor.$$

Exploiting the telescoping nature of the Multiplier Property (shown in Equation (2)) and summing from $i = n+1$ to m, we get

$$(b_n - d_m) + \frac{1}{3}\left(\lfloor \alpha m + t\rfloor - \lfloor \alpha n + t\rfloor\right) = \left\lfloor \frac{\alpha m + 3t'}{3} \right\rfloor - \left\lfloor \frac{\alpha n + 3t'}{3} \right\rfloor. \tag{3}$$

Since $\alpha \notin \mathbb{Q}$, we can pick n, m so that

$$\frac{\alpha m + 3t'}{3} = k_m + \varepsilon_m \qquad \text{and} \qquad \frac{\alpha n + 3t'}{3} = k_n - \varepsilon_n$$

where $k_m, k_n \in \mathbb{Z}$ and $\varepsilon_m, \varepsilon_n$ are arbitrarily small positive numbers. Upon this choice, the right side of Equation (3) simplifies to $k_m - k_n + 1$. By rearranging and substituting into Equation (3), we get

$$\lfloor 3(k_m + \varepsilon_m) + (t - 3t')\rfloor - \lfloor 3(k_n - \varepsilon_n) + (t - 3t')\rfloor = 3(k_m - k_n) + 3 - 3(b_n - d_m),$$

but since $b_n - d_m$ is bounded above by $2/3$, we conclude

$$\lfloor 3(k_m + \varepsilon_m) + (t - 3t')\rfloor - \lfloor 3(k_n - \varepsilon_n) + (t - 3t')\rfloor \geq 3(k_m - k_n) + 1. \tag{4}$$

If $(t - 3t' \bmod 1) = \gamma \neq 0$, choosing $\varepsilon_n, \varepsilon_m \ll \gamma$ gives a contradiction (with the left hand side of Equation (4) yielding $3(k_m - k_n)$). Thus, $t = 3t' \bmod 1$.

Case $\lambda = 2$: Since this case is nearly identical to the $\lambda = 1/3$ case, we will omit the details, noting only that in this case the relationship between bottom labels and top labels is reversed. That is, in this case fix $\alpha = \alpha_{k+1}$ and rewrite $\alpha_k = \alpha/2$.

We have now shown that $(\vec{\alpha}, \vec{t})$ satisfies the BC property. To complete the proof and show that y arises as a Basic Construction, we need to show that every Sturmian sequence in $\Phi(y)$ can be written with exclusively $\mathbf{R}_{\lfloor\cdot\rfloor}$ or exclusively $\mathbf{R}_{\lceil\cdot\rceil}$.

Notice that since every component of $\vec{\alpha}$ is rationally related to α_0, we have either $\vec{\alpha} \in \mathbb{Q}^{\mathbb{Z}}$ or $\vec{\alpha} \in (\mathbb{Q}^c)^{\mathbb{Z}}$. By assumption however, $\vec{\alpha} \in (\mathbb{Q}^c)^{\mathbb{Z}}$ and so $n\vec{\alpha} + \vec{t} \in \mathbb{Q}^{\mathbb{Z}}$ for at most one n. Define

$$r_{i,n} = n\alpha_i + t_i.$$

Suppose that the ith row of $\Phi(y)$ requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$ to be written as a rotation sequence. This implies that for some n we have $r_{i,n} \in \mathbb{Z}$. Fix this n. By our observation that $m\vec{\alpha} + \vec{t} \in \mathbb{Q}^{\mathbb{Z}}$ for at most one m, we may conclude that $r_{j,n'} \notin \mathbb{Z}$ for any $n' \neq n$ and $j \in \mathbb{Z}$.

Let

$$B = \{i : (\Phi(y))_i \text{ requires } \mathbf{R}_{\lfloor\cdot\rfloor} \text{ or } \mathbf{R}_{\lceil\cdot\rceil}\},$$

and notice again that by the uniqueness of n (with n still being fixed as above), $B = \{j : r_{j,n} \in \mathbb{Z}\}$. We will now show that B consists of a contiguous sequence of integers.

Exploiting the fact that $(\vec{\alpha}, \vec{t})$ satisfies the BC property, we may conclude $r_{j+1,n} = 2r_{j,n}$ or $r_{j+1,n} = \frac{1}{3}(r_{j,n} + i)$ where $i \in \{0, 1, 2\}$. This implies that if $|r_{j,n}|_3 > 0$ then $|r_{j+1,n}|_3 > 0$ where $|\cdot|_3$ is the 3-valuation.

If $b \in B$ and $b + 1 \notin B$, this means $r_{b,n} \in \mathbb{Z}$ but $r_{b+1,n} \notin \mathbb{Z}$ and so $|r_{b+1,n}|_3 > 0$ (since multiplying by 2 keeps us in $\mathbb{Z}$, the only way to leave $\mathbb{Z}$ is to divide by 3). However, $|r_{b+1,n}|_3 > 0$ implies that $r_{b+i,n} \notin \mathbb{Z}$ for all $i > 0$. We conclude that B cannot contain any gaps.

Since B consists of a contiguous set of integers, it will complete the proof if we show that there do not exist two adjacent rows in $\Phi(y)$ where one requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the other requires $\mathbf{R}_{\lceil\cdot\rceil}$. We will conclude the proof by showing that the rules of the Kari-Culik tiling forbid such an occurrence.

Suppose $s = \mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t) \neq \mathbf{R}_{\lceil\cdot\rceil}(\alpha, t) = s'$ and $\alpha \notin \mathbb{Q}$, and notice s and s' differ only by a transposition of two adjacent coordinates. For simplicity, assume s and s' differ at coordinates 1 and 2 and that $\alpha(s) \in [0, 1]$ so that s and s' consist of the symbols 0 and 1. We then have

$$s = \cdots s_0 s_1 s_2 s_3 \cdots \qquad \text{and} \qquad s' = \cdots s_0 s_2 s_1 s_3 \cdots,$$

and in particular $s_1 \neq s_2$. Since s and s' are both valid Sturmian sequences, we may conclude that $s_0 = s_3$, since if $s_0 \neq s_3$ either s or s' would contain both a $1, 1$ and a $0, 0$ (which is impossible in a Sturmian sequence [4]). Further, since s requires $\lfloor\cdot\rfloor$, we know $s_1 > s_2$.

In general, we will call a length four word $w_{\alpha,t} = w_0 w_1 w_2 w_3$ or $w_{\alpha,t} = w_0 w_2 w_1 w_3$ a *straddle word* of a Sturmian sequence if

$$w_0 w_1 w_2 w_3 = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_i^{i+3} \qquad \text{and} \qquad w_0 w_2 w_1 w_3 = (\mathbf{R}_{\lceil\cdot\rceil}(\alpha, t))_i^{i+3}$$

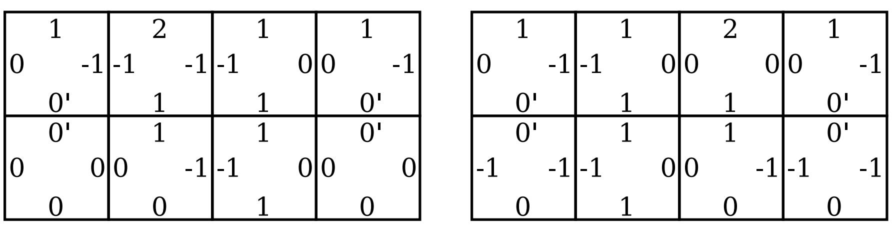
for some i (or vice versa) and $w_1 \neq w_2$. The previous argument shows that if $w_{\alpha,t}$ is a straddle word, then $w_0 = w_3$. It also shows that if a Sturmian sequence requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$, it necessarily contains a straddle word.

Now, consider a row r of y where the sequence of bottom labels requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the top labels require $\mathbf{R}_{\lceil\cdot\rceil}$ (or vice versa) and let w^t and w^b be the straddle words for the labels on the top of r and the bottom of r respectively. Since the top sequence requires $\mathbf{R}_{\lfloor\cdot\rfloor}$ and the bottom sequence requires $\mathbf{R}_{\lceil\cdot\rceil}$, we conclude that $w_1^t > w_2^t$ and $w_1^b < w_2^b$ (or vice versa if the roles of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are reversed). We will call a pair of straddle words like these, whose middle two symbols satisfy opposite inequalities, *misaligned*.

We will complete the proof by observing that misaligned straddle words cannot occur in y.

By enumerating all pairs of length-four words (w^t, w^b) that arise as tops and corresponding bottoms of rows of type 2.1, we see that out of the 64 possibilities, only four have the property that $w_1^t \neq w_2^t$ and $w_1^b \neq w_2^b$ (which is necessary to be a straddle word). Out of those four, none are misaligned. Similarly, in a row of type $\frac{1}{3}$, of the 96 possibilities, 24 differ in their middle symbols and out of those, none are misaligned straddle words.

Now consider a row of type 2.2. Out of the 128 possible sequences of length 4, there are exactly two ways to obtain misaligned straddle words, namely:



This gives misaligned straddle pairs of $(w^t, w^b) = (1211, 0010)$ and $(w^t, w^b) = (1121, 0100)$. Since we are considering a type 2.2 row, the Sturmian angle for the sequence of bottoms must be in $[1/3, 1/2]$. Thus, there cannot be three 0's in a row. We therefore conclude the symbol before the word $w^b = 0010$ must be a 1 and the symbol after the word $w^b = 0100$ must be a 1. Since we are considering 0010 and 0100 as straddle subwords of some pair of Sturmian sequences and these Sturmian sequences must agree everywhere except for a single transposition of symbols, we conclude $w^b = 0010$ and $w^b = 0100$ must be subwords of 100101 and 101001. By a similar argument, the top straddle words must be subwords of 211212 and 212112. Thus, the stacked tile to the left or right of the designated blocks must have a bottom label of 1 and a top label of 2. Inspecting the two stacked tiles with this property reveals that neither of them are compatible with the potential misaligned straddle words shown, and thus misaligned straddle words cannot appear in y. ■

4 The Subset KC

Recall that KC is the subset of Kari-Culik tilings whose bottom labels form generalized Sturmian sequences. The following theorem allows us to focus on the Sturmian sequences made from bottom labels of KC as opposed to configurations on $\mathfrak{K}^{\mathbb{Z}^2}$.

Theorem 10. *$\Phi|_{KC_{\mathbb{Q}^c}}$ is one-to-one and Φ is at most sixteen-to-one.*

Proof. Fix $x \in KC$ and consider a row r of x. Let r^t be the Sturmian sequence formed by the top labels of r and let r^b be the Sturmian sequence formed by the bottom labels of r. Further, let $\alpha^t = \alpha(r^t)$ and $\alpha^b = \alpha(r^b)$, and if $w^t = (r^t)_i^j$ is a subword of r^t, then $w^b = (r^b)_i^j$ is the corresponding subword of r^b. That is w^b is the subword of r^b whose indices are identical to the indices of w^t. Note that give a w^t absent of the indices from which it came, w^b may not be uniquely defined.

Notice that by the multiplier property (Equation (1)), r is uniquely determined by (r^t, r^b) and a single left label of one of the tiles in r.

Define Φ_2 by $\Phi_2(r) = (r^t, r^b)$ and extend Φ_2 to work on subwords of r. By our previous observation, if there is a subword $r' \subset r$ such that $\Phi_2^{-1}(\Phi_2(r'))$ contains a single element, then r is uniquely determined by (r^t, r^b). We will show that if $x \in KC_{\mathbb{Q}^c}$, this is always the case. Moving forward, we analyze r separately depending on its type.

Case r is of type $\frac{1}{3}$: In this case, we know $\alpha^t \in [1/3, 2/3]$ and $\alpha^b \in [1, 2]$.

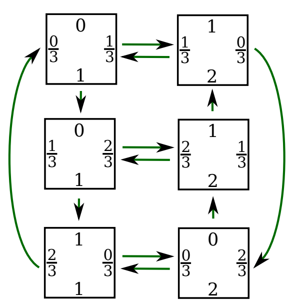


Figure 2: Transition graph for type $\frac{1}{3}$ tiles.

Figure 2 shows the transition graph moving left to right in a type $\frac{1}{3}$ row. Notice that there is only one way for 11 or 00 to appear as top labels in a type $\frac{1}{3}$ row. In particular, if $w^t = 11$ then $w^b = 22$ and if $w^t = 00$ then $w^b = 11$ and $|\Phi_2^{-1}(11, 22)| = |\Phi_2^{-1}(00, 11)| = 1$. Thus, if r^t contains the word 11 or 00, r is uniquely determined by (r^t, r^b).

If r^t contains neither 11 nor 00, then $\alpha^t = 1/2$ and $r^t = \cdots 101010 \cdots$. Analysing further, if $w^t = 01$ then $w^b \in \{12, 21, 11, 22\}$. We note that $|\Phi_2^{-1}(01, 11)| = |\Phi_2^{-1}(01, 22)| = |\Phi_2^{-1}(01, 21)| = 1$ and $|\Phi_2^{-1}(01, 12)| = 2$. Thus, Φ on a row of type $\frac{1}{3}$ is only non-unique if $\alpha^t = 1/2$ which further implies $\alpha^b = 3/2$.

Case r is of type 2.1: In this case, we know $\alpha^t \in [1, 2]$ and $\alpha^b \in [1/2, 1]$.

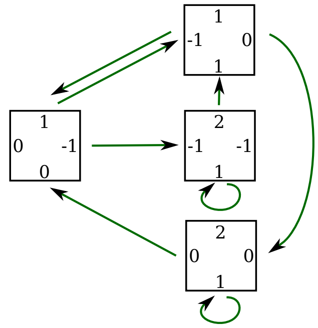


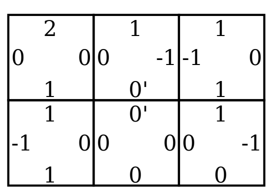
Figure 3: Transition graph for a type 2.1 row.

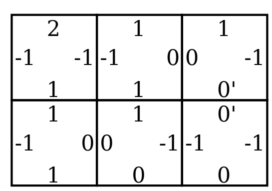
Figure 3 shows the transition graph moving left to right in a type 2.1 row. Notice that there is only one way a type 2.1 row can contain 0 as a bottom label. This means r is uniquely determined by (r^t, r^b) unless $\alpha^b = 1$ (since $\alpha^b < 1$ implies a zero occurs in r^b) and consequently $r^b = \cdots 111 \cdots$. If $w^b = 1$ then $w^t \in \{1, 2\}$ with $|\Phi_2^{-1}(1,1)| = 1$ and $|\Phi_2^{-1}(2,1)| = 2$.

Case r is of type 2.2: In this case, we know $\alpha^t \in [4/3, 2]$ and $\alpha^b \in [1/3, 1/2]$ (since we interpret our multiplier as 4 in this case).

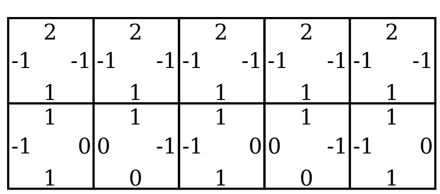
Because $\alpha^b \in [1/3, 1/2]$, we know that r^b cannot contain 11 (since if a rotation sequence contains 11, then its angle must be strictly larger than 1/2). Therefore, it must contain 100 or 10101 as a subword. Further, if $\alpha^b \in (1/3, 1/2)$, r^b must contain 10100 as a subword. Let $w^b \in \{100, 10101, 10100\}$. Below is a list of all pairs (w^t, w^b) such that $|\Phi_2^{-1}(w^t, w^b)| > 1$.

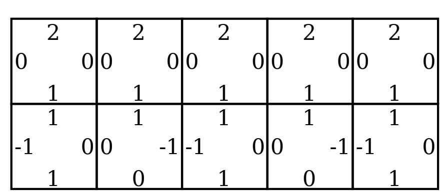
$(w^t, w^b) = (211, 100)$, which can be obtained in exactly two ways, namely



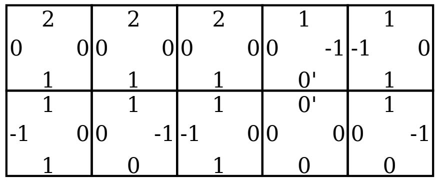


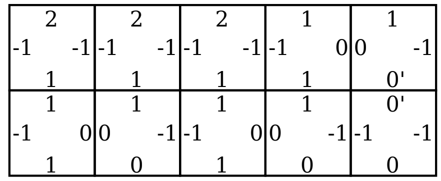
$(w^t, w^b) = (22222, 10101)$, which can be obtained in exactly two ways, namely





and $(w^t, w^b) = (22211, 10100)$, which can be obtained in exactly two ways, namely





Considering the pair $(w^t, w^b) = (22211, 10100)$, we see that 22211 is not a generalized Sturmian sequence (since it contains both 11 and 22 as subwords), and so this situation never occurs. This means if Φ_2 is not invertible, $\alpha^b = 1/3$, which corresponds to the first case, or $\alpha^b = 1/2$ which corresponds to the second case.

As a result of our case-by-case analysis, we see that if $\alpha^b \notin \{1/3, 1/2, 1, 3/2\}$, then r is uniquely determined and so when restricted to $KC_{\mathbb{Q}^c}$, Φ is one-to-one. Further, since for $\alpha^b \in \{1/3, 1/2, 1, 3/2\}$ we have Φ_2 is at most two-to-one, we conclude in general that Φ is at most sixteen-to-one (since f prevents α^b from taking any particular value in $\{1/3, 1/2, 1, 3/2\}$ more than once). ■

In light of Theorem 10, we will treat points in KC and points in $\Phi(KC)$ interchangeably, differentiating only when needed.

Recall that $T : KC \to KC$ is the horizontal shift and $S : KC \to KC$ is the vertical shift.

Proposition 11. *For $x \in KC$, we have*

$$\alpha(Sx) = f(\alpha(x)),$$

where f is applied component-wise and if $x \in KC_{\mathbb{Q}^c}$,

$$t(Tx) = t(x) + \alpha(x).$$

Proof. This immediately follows from Corollary 5 and the definition of a rotation sequence. ■

We will now produce a parameterization of points in $KC_{\mathbb{Q}^c}$ in a similar fashion to the way a Sturmian sequence may be parameterized by an angle, phase, and choice of floor or ceiling function.

Definition. *Let*

$$\mathcal{T} = \varprojlim \mathbb{R}/(6^n\mathbb{Z})$$

be the inverse limit of the groups $\mathbb{R}/(6^n\mathbb{Z})$ as $n \to \infty$.

We view a point $t = (t_0, t_1, \ldots) \in \mathcal{T}$ as a sequence of real numbers endowed with the product topology and satisfying the consistency condition $t_i = t_{i+1} \bmod 6^i$ with $\mathrm{proj}_k(t) = t_k$. As such, we can define scalar-multiplication functions M_a and $M_{1/a}$ for $a \in \{2, 3\}$ as follows:

$$M_a(t_0, t_1, \ldots) = (at_0 \bmod 1, at_1 \bmod 6, at_2 \bmod 6^2, \ldots)$$

and

$$M_{1/a}(t_0, t_1, \ldots) = M_{6/a}(t_1/6, t_2/6, t_3/6, \ldots).$$

Proposition 12. *$M_2, M_3, M_{1/2}, M_{1/3} : \mathcal{T} \to \mathcal{T}$ are bijective homomorphisms.*

The proof of Proposition 12 is straightforward, relying on the fact that 2 and 3 divide 6. From now on, for $t \in \mathcal{T}$ we may write at or t/a instead of $M_a t$ and $M_{1/a}t$. Further, for $r \in \mathbb{R}$, we may define scalar addition $A_r : \mathcal{T} \to \mathcal{T}$ by

$$A_r(t_0, t_1, \ldots) = (t_0 + r \bmod 1, t_1 + r \bmod 6, t_2 + r \bmod 6^2, \ldots)$$

and we may write $t + r$ instead of $A_r(t)$.

Notation. *Extend f to a function $\hat{f} : [1/3, 2] \times \mathcal{T} \to [1/3, 2] \times \mathcal{T}$ by*

$$\hat{f}(\alpha, t) = \begin{cases} (2\alpha, 2t) & \text{if } \alpha \in [1/3, 1) \\ (\alpha/3, t/3) & \text{if } \alpha \in [1, 2] \end{cases}.$$

Notice that $\hat{f} : [1/3, 2] \times \mathcal{T} \to [1/3, 2] \times \mathcal{T}$ is a bijection. Morally, we will show that $[1/3, 2] \times \mathcal{T}$ is a parameterization of KC. However, since trouble arises for generalized Sturmian sequences with rational angles and Sturmian sequences whose phase vector contains zero, we focus our attention to the sets $([1/3, 2]\backslash\mathbb{Q}) \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ and $KC_{\mathbb{Q}^c}$.

Lemma 13. *Let $\mathcal{O}_1$ be the orbit of 1 under f, and let*

$$X = \{(\vec{\alpha}, \vec{t}) : (\vec{\alpha}, \vec{t}) \text{ satisfies the BC property and } 1 \notin \vec{\alpha}\}.$$

There exists a bijection $W : ([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T} \to X$ that respects the dynamical relationships. That is, for $(\alpha, t) \in ([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T}$ such that $W(\alpha, t) = (\vec{\alpha}, \vec{t})$, we have

$$W(\alpha, t + \alpha) = (\vec{\alpha}, \vec{t} + \vec{\alpha} \bmod 1) \qquad \text{and} \qquad W \circ \hat{f}(\alpha, t) = (\sigma(\vec{\alpha}), \sigma(\vec{t})).$$

Proof. Defining W is straightforward. Fix $(\alpha, t) \in ([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T}$ and define $W(\alpha, t) = (\vec{\alpha}, \vec{t})$ where $(\alpha_i, t_i) = (\mathrm{id} \times \mathrm{proj}_0) \circ \hat{f}^i(\alpha, t)$. The definition of $\hat{f}$ acting on $([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T}$ ensures that $(\vec{\alpha}, \vec{t})$ satisfies the BC property and respects the desired dynamical relationships.

W is clearly one-to-one in the first coordinate. Further, if $t, t' \in \mathcal{T}$ with $t \neq t'$, we have that $\mathrm{proj}_j t \neq \mathrm{proj}_j t'$ for some j. From this we deduce that $\mathrm{proj}_0 \circ \hat{f}^{j'}(t) \neq \mathrm{proj}_0 \circ \hat{f}^{j'}(t')$ for some $j' \geq j$, and so W is one-to-one. After the construction of W^{-1}, it will be evident that it is onto.

Constructing W^{-1} is slightly more difficult. Fix $(\vec{\alpha}, \vec{t}) \in X$. Let $\Lambda_i = \alpha_i/\alpha_0$. Let $|\cdot|_2$ and $|\cdot|_3$ be the 2-valuation and 3-valuation respectively, and note that Λ_i is always rational and so its 2 and 3 valuations are defined.

Let $j(i) = \min\{n \geq 0 : |\Lambda_n|_3 = i\}$ and define the subsequences $(t_i^{(3)})_{i=0}^\infty$ and $(\Lambda_i^{(3)})_{i=0}^\infty$ of $(t_i)_{i=0}^\infty$ and $(\Lambda_i)_{i=0}^\infty$ by

$$t_i^{(3)} = t_{j(i)} \qquad \text{and} \qquad \Lambda_i^{(3)} = \Lambda_{j(i)}.$$

Similarly, let $j'(i) = \min\{n \geq 0 : |\Lambda_{-n}|_2 = i\}$ and define the subsequences $(t_i^{(2)})_{i=0}^\infty$ and $(\Lambda_i^{(2)})_{i=0}^\infty$ by $t_i^{(2)} = t_{-j'(i)}$ and $\Lambda_i^{(2)} = \Lambda_{-j'(i)}$. We've constructed $t^{(2)}$ and $t^{(3)}$ to be the values of $\vec{t}$ corresponding to where we "divide α by 2" or "divide α by 3" respectively.

To construct $W^{-1}(\vec{\alpha}, \vec{t})$, first consider the point $z^{(3)} = (z_0^{(3)}, z_1^{(3)}, \dots) \in \varprojlim \mathbb{R}/(3^n\mathbb{Z})$ defined inductively in the following way. Let $z_0^{(3)} = t_0^{(3)} = t_0^{(2)}$. Fix j and suppose for all $i < j$ we have

$$\Lambda_i^{(3)} z_i^{(3)} = t_i^{(3)} \bmod 1.$$

Let p be such that $\Lambda_j^{(3)} = \frac{2^p}{3}\Lambda_{j-1}^{(3)}$. We now have that $3t_j^{(3)} = 2^p t_{j-1}^{(3)} \bmod 1$ and so along with our induction hypothesis there exist unique $r', r'' \in \{0, 1, 2\}$ so

$$3t_j^{(3)} = 2^p t_{j-1}^{(3)} + r' \bmod 3$$
$$\Lambda_{j-1}^{(3)} z_{j-1}^{(3)} - t_{j-1}^{(3)} = r'' \bmod 3.$$

Define $z_j^{(3)} = z_{j-1}^{(3)} + r3^{j-1}$ where $r \in \{0, 1, 2\}$ is the unique solution to $2^p r'' - r' + r2^p\Lambda_{j-1}^{(3)} 3^{j-1} = 0 \bmod 3$. Note that since $2^p\Lambda_{j-1}^{(3)} 3^{j-1} \in \mathbb{Z}$ and contains no multiples of 3, such an r always exists. By construction, $z_j^{(3)}$ satisfies $z_{j-1}^{(3)} = z_j^{(3)} \bmod 3^{j-1}$. We will now verify that $\Lambda_j^{(3)} z_j^{(3)} - t_j^{(3)} = 0 \bmod 1$.

Substituting, we see

$$\Lambda_j^{(3)} z_j^{(3)} - t_j^{(3)} = \tfrac{2^p}{3}\Lambda_{j-1}^{(3)} z_j^{(3)} - t_j^{(3)} = \tfrac{2^p}{3}\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - t_j^{(3)}.$$

Finally, multiplying by 3, we have

$$2^p\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - 3t_j^{(3)} = 2^p\Lambda_{j-1}^{(3)}(z_{j-1}^{(3)} + r3^{j-1}) - 2^p t_{j-1}^{(3)} - r' \bmod 3$$
$$= 2^p\left(\Lambda_{j-1}^{(3)} z_{j-1}^{(3)} - t_{j-1}^{(3)} + r\Lambda_{j-1}^{(3)} 3^{j-1}\right) - r' = 2^p r'' - r' + r2^p\Lambda_{j-1}^{(3)} 3^{j-1} = 0 \bmod 3$$

as desired.

We have shown that $z^{(3)}$ exists and is unique. In an analogous way, construct $z^{(2)} \in \varprojlim \mathbb{R}/(2^n\mathbb{Z})$. Finally, since $z_0^{(3)} = z_0^{(2)}$, by the Chinese remainder theorem we may produce $z = (z_0, z_1, \dots) \in \mathcal{T}$ such that $z_i^{(3)} = z_i \bmod 3^i$ and $z_i^{(2)} = z_i \bmod 2^i$.

It is worth noting now that by construction, z_i is the unique simultaneous solution to

$$\Lambda_i^{(3)} z_i = t_i^{(3)} \bmod 1$$
$$\Lambda_i^{(2)} z_i = t_i^{(2)} \bmod 1$$

where $z_i \in \mathbb{R}/(6^i\mathbb{Z})$ and $z_{i-1} = z_i \bmod 6^{i-1}$.

Having established existence and uniqueness of z, we may define $W^{-1}(\vec{\alpha}, \vec{t}) = (\alpha_0, z)$. The fact that W^{-1} is an inverse is now immediate by construction: if $|\Lambda_j|_3 = k \geq 0$, then $\Lambda_j \mathrm{proj}_k(z) = t_j^{(3)} \bmod 1$, and if $|\Lambda_j|_2 = k \geq 0$, then $\Lambda_j \mathrm{proj}_k(z) = t_{-j}^{(2)} \bmod 1$. ■

We restricted the domain of W to $([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T}$ instead of $[1/3, 2] \times \mathcal{T}$ because the function f could be defined so that $f(1) = 1/3$ or $f(1) = 2$, and both ways would be consistent with the rules of the Kari-Culik tilings. As such, this presents a small obstruction to W being a bijection on $[1/3, 2] \times \mathcal{T}$.

To allow for a clearer statement of Proposition 14, we will extend notation so that $\hat{f}(\alpha, t, R) = (\alpha', t', R)$ where $(\alpha', t') = \hat{f}(\alpha, t)$.

Proposition 14. *There exists maps* $K : [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\} \to KC$ *and* $K' : KC_{\mathbb{Q}^c} \to [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ *so that* $K \circ K' = \mathrm{id}$ *and for any* $y \in KC_{\mathbb{Q}^c}$*,* K' *satisfies the following relationships:*

$$\begin{aligned} K'(y) &= (\alpha, t, R) \\ K'(Ty) &= (\alpha, t+\alpha, R) \\ K'(Sy) &= \hat{f}(\alpha, t, R) \end{aligned}$$

for some $(\alpha, t, R) \in [1/3, 2] \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$.

Proof. This proposition is a corollary of Lemma 13.

Let $X = \{(\vec{\alpha}, \vec{t}) : (\vec{\alpha}, \vec{t})$ satisfies the BC property$\}$. Observe that by Proposition 9, we have explicit maps $A : X \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\} \to KC$ and $A' : KC_{\mathbb{Q}^c} \to X \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}$ so that $A \circ A' = \mathrm{id}$. Projecting onto the first coordinate, we also have if $A'(x) = (\vec{\alpha}, \vec{t})$, then $A'(Tx) = (\vec{\alpha}, \vec{t} + \vec{\alpha} \mod \vec{1})$ and $A'(Sx) = (\sigma(\vec{\alpha}), \sigma(\vec{t}))$, where σ is the shift on bi-infinite vectors.

Lemma 13 gives us an invertible map $W : ([1/3, 2]\backslash\mathcal{O}_1) \times \mathcal{T} \to X$ that respects the dynamical relationships. Without changing notation, extend W (non-bijectively) to a map $W : [1/3, 2] \times \mathcal{T} \to X$ in a way that respects the dynamical relationships. We may now define $K = A \circ W$. Finally, since the first component of the image of A' avoids $\mathcal{O}_1$, we may define $K' = W^{-1} \circ A'$. ∎

Where convenient, we will think of $K : [1/3, 2] \times \mathcal{T} \to KC$ and $K' : KC_{\mathbb{Q}^c} \to [1/3, 2] \times \mathcal{T}$ without the notational burden of keeping track of the choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ or $\mathbf{R}_{\lceil\cdot\rceil}$. Further, since points in $KC_{\mathbb{Q}^c}$ requiring $\mathbf{R}_{\lceil\cdot\rceil}$ to be expressed may be approximated by points in $KC_{\mathbb{Q}^c}$ requiring only $\mathbf{R}_{\lfloor\cdot\rfloor}$ (and vice versa), the choice of $\mathbf{R}_{\lfloor\cdot\rfloor}$ and $\mathbf{R}_{\lceil\cdot\rceil}$ may be safely ignored. Considering the relationships outlined in the proof of Proposition 14, we may now deduce the following theorem.

Theorem (Theorem A)**.** $KC_{\mathbb{Q}^c}$ *is a skew product. That is, if* K' *is the map defined in Proposition 14 and* $y \in KC_{\mathbb{Q}^c}$*, we have the following relationships:*

$$\begin{aligned} K'(y) &= (\alpha, t) \\ K'(Ty) &= (\alpha, t+\alpha) \\ K'(Sy) &= \hat{f}(\alpha, t). \end{aligned}$$

5 Minimality of KC

In light of Theorem A, we will first consider the minimality of the $\mathbb{Z}^2$ action of $(\hat{T}, \hat{S})$ on $[1/3, 2] \times \mathcal{T}$ where $\hat{T}$ and $\hat{S}$ are defined by

$$\hat{T}(\alpha, t) = (\alpha, t+\alpha) \qquad \text{and} \qquad \hat{S}(\alpha, t) = \hat{f}(\alpha, t).$$

Though the minimality of $([1/3,2]\times\mathcal{T},\hat{T},\hat{S})$ and (KC,T,S) can be deduced from the explicit methods in Section 6, we present an abstract proof of minimality that may be of independent interest.

Since trouble arises when considering (α,t) with $\alpha\in\mathbb{Q}$, we will use the Uryson Metrization Theorem to produce a metric $\hat{d}$ such that $([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}$, endowed with the subspace topology, is a complete metric space with respect to $\hat{d}$.

Proposition 15. *The $\mathbb{Z}^2$ action of $(\hat{T},\hat{S})$ on $([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}$ is minimal with respect to $\hat{d}$.*

Proof. We first claim that since $\alpha\notin\mathbb{Q}$, the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense in $\mathcal{T}$. For any k, it is clear that the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense modulo 6^k. That is, the second coordinate of $\mathrm{id}\times\mathrm{proj}_k(\mathcal{O}_{\hat{T}}(\alpha,t))$ is dense in $\mathrm{proj}_k(\mathcal{T})$. We therefore have the second coordinate of $\mathcal{O}_{\hat{T}}(\alpha,t)$ is dense modulo 6^i for any $i\le k$. Denseness now follows from the definition of the product topology on $\mathcal{T}$.

Using this observation, it is clear that for any $(\alpha,t'),(\alpha,t)\in([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}$, we have $(\alpha,t)\in\overline{\mathcal{O}_{\hat{T}}(\alpha,t')}$.

To complete the proof, fix $(\alpha',t'),(\alpha,t)\in([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}$. Since the orbit of any point under $f:[1/3,2]\to[1/3,2]$ is dense, we may find a point $(\alpha,t'')\in\overline{\mathcal{O}_{\hat{S}}(\alpha',t')}$. Using our previous observation, $(\alpha,t)\in\overline{\mathcal{O}_{\hat{T}}(\alpha,t'')}$. This means $(\alpha,t)\in\overline{\mathcal{O}(\alpha',t')}$, and so the orbit of every point is dense. ■

If the function $K:([1/3,2]\backslash\mathbb{Q})\times\mathcal{T}\to KC_{\mathbb{Q}^c}$ were continuous, this would give us a quick proof of the minimality of $KC_{\mathbb{Q}^c}$. Unfortunately this is not the case, but the set of points where K is continuous is a dense G_δ.

Proposition 16. *The set of points of continuity of $K:[1/3,2]\times\mathcal{T}\to KC$ is a dense residual set G and $K(G)$ dense in KC.*

Proof. Let $A_{m\times n}=\{0,\dots,m-1\}\times\{0,\dots,n-1\}$. Suppose $E\subset[1/3,2]\times\mathcal{T}$ is an open set on which K is not continuous. Then, for some $m,n\in\mathbb{N}$, E must contain a point (α,t) so that $K(\alpha,t,\mathbf{R}_{\lfloor\cdot\rfloor})|_{A_{m\times n}}\neq K(\alpha,t,\mathbf{R}_{\lceil\cdot\rceil})|_{A_{m\times n}}$. Any point (α,t) that satisfies this is a point of discontinuity, and any point of discontinuity satisfies this condition for some m,n.

Define

$$B_{m\times n}=\{(\alpha,t):K(\alpha,t,\mathbf{R}_{\lfloor\cdot\rfloor})|_{A_{m\times n}}\neq K(\alpha,t,\mathbf{R}_{\lceil\cdot\rceil})|_{A_{m\times n}}\},$$

and notice that since the only points of discontinuity of $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ are $\mathbb{Z}$, $B_{m\times n}$ is a closed, nowhere dense set implying that $B^c_{m\times n}$ is a dense open set. Let

$$G=\bigcap_{m,n\in\mathbb{N}}B^c_{m\times n}.$$

We now have that by construction, the set of continuity points of K is the dense residual set G. Further, since $K(B^c_{m\times n})|_{A_{m\times n}}$ contains every $m\times n$ configuration that appears in KC, $K(G)$ is dense in KC. ■

Corollary 17. *If $O \subset KC$ is a non-empty open set, then $K^{-1}(O)$ contains a non-empty open set.*

Proof. Let G be a dense set of points of continuity of K whose image is dense. Fix an open set $O \subset KC$. Since $K(G)$ is dense, $K(G) \cap O \neq \emptyset$ and so O is a neighborhood of the image of a point of continuity of K. Thus $K^{-1}(O)$ is a neighborhood and so contains an open set. ■

Lemma 18. *Suppose (X, σ) and $(Y, \hat{\sigma})$ are dynamical systems and that $g : X \to Y$ is a surjective map that satisfies $g \circ \sigma = \hat{\sigma} \circ g$. If the set of points of continuity of g is a dense set G and $g(G)$ is dense in Y, then (X, σ) minimal implies $(Y, \hat{\sigma})$ minimal.*

Proof. We will first show that if $D \subset X$ is dense, then $g(D)$ is dense. Fix $y \in Y$ and some neighborhood N_y of y. Let G be a dense set of points of continuity for g such that $g(G)$ is dense. Then, $N_y \cap g(G) \neq \emptyset$ and so N_y is a neighborhood of the image of a point of continuity. Thus $g^{-1}(N_y)$ is a neighborhood of some point. Since D is dense, $D \cap g^{-1}(N_y) \neq \emptyset$, and so $g(D)$ intersects every neighborhood and must be dense.

To complete the proof, fix $y \in Y$, and by surjectivity of g find $x \in X$ so $g(x) = y$. Suppose X is minimal. We have that $\mathcal{O}x$ is dense, and so $g(\mathcal{O}x) = \mathcal{O}g(x) = \mathcal{O}y$ is dense. ■

Proposition 19. $\Phi(KC) = \overline{\Phi(KC_{\mathbb{Q}^c})}$.

Proof. Fix $y \in \Phi(KC)$, $A = \{0, 1, \ldots, m-1\} \times \{0, 1, \ldots, n-1\}$, and the cylinder set $C = \{x \in \Phi(KC) : x|_A = y|_A\}$. C is open and so by Corollary 17, $K^{-1}(C)$ contains an open set. Thus, there is some $(\alpha, t) \in K^{-1}(C)$ where $\alpha \notin \mathbb{Q}$. ■

Theorem (Theorem B)**.** *KC is minimal with respect to the group action of $\mathbb{Z}^2$ by translation.*

Proof. We will first show that every orbit of every point in $KC_{\mathbb{Q}^c}$ is dense in KC', the subset of KC where Φ is one-to-one, and then that the orbit closure of any point in KC intersects $KC_{\mathbb{Q}^c}$. Finally we will show that KC' is dense in KC. Here we must take special care to differentiate between KC and $\Phi(KC)$.

By Proposition 14, we have that

$$K\Big(([1/3, 2]\backslash\mathbb{Q}) \times \mathcal{T} \times \{\mathbf{R}_{\lfloor\cdot\rfloor}, \mathbf{R}_{\lceil\cdot\rceil}\}\Big) = KC_{\mathbb{Q}^c}.$$

Now, by Proposition 16, K satisfies the conditions of Lemma 18. Thus, since Proposition 15 states that $([1/3, 2]\backslash\mathbb{Q}) \times \mathcal{T}$ is minimal with respect to $(\hat{T}, \hat{S})$, Lemma 18 gives us that the orbit of every point in $KC_{\mathbb{Q}^c}$ is dense in $KC_{\mathbb{Q}^c}$.

Let

$$KC' = \{x \in KC : \alpha(x) \text{ does not contain } 1/3, 1/2, 1, \text{ or } 3/2\}.$$

and recall that the proof of Theorem 10 shows that Φ is one-to-one exactly on KC'. Thus, since $\Phi(KC_{\mathbb{Q}^c})$ is dense in $\Phi(KC)$, by continuity of Φ, we may conclude that the orbit of any point in $KC_{\mathbb{Q}^c}$ is dense in KC'.

Next, we will show that for any $y \in KC$, we have $\overline{\mathcal{O}y} \cap KC_{\mathbb{Q}^c} \neq \emptyset$. Fix $y \in KC$ and let $\alpha(y) = (\ldots, \alpha_0, \alpha_1, \ldots)$. Choose f to be the function

$$f(x) = \begin{cases} 2x & \text{if } x \in [1/3, 1) \\ x/3 & \text{if } x \in [1, 2] \end{cases}$$

or

$$f(x) = \begin{cases} 2x & \text{if } x \in [1/3, 1] \\ x/3 & \text{if } x \in (1, 2] \end{cases}$$

such that $f(\alpha_i) = \alpha_{i+1}$. Since the orbit of every point under f is dense, we may find $y' \in \overline{\mathcal{O}y}$ so that $\alpha(y')$ contains only irrationals. However, since $\alpha(y')$ contains only irrationals, $y' \in KC_{\mathbb{Q}^c}$ and so $\mathcal{O}(y')$ is dense in $KC_{\mathbb{Q}^c}$ showing that $\mathcal{O}y$ is dense in $KC_{\mathbb{Q}^c}$.

To complete the proof, we will now show KC' is dense in KC. We will do this by considering cases. Suppose $y \in KC \backslash KC'$. This means for some j, $\alpha((y)_j) \in \{1/3, 1/2, 1, 3/2\}$. For simplicity, assume this occurs at $j = 0$ and let $\alpha = \alpha((y)_0)$.

Case $\alpha = 3/2$: If $\alpha = 3/2$, the sequence of bottom labels must be $\cdots 121212 \cdots$. Looking at the transition graph in Figure 2, a sequence to bottom labels of $\cdots 121212 \cdots$ can be realized in two ways. Call the configuration using the tiles in the bottom of the diagram configuration A and the configuration using the tiles in the top of the diagram configuration B, and notice that if $\alpha = 3/2 + \delta$ for $0 < \delta$ small, then the row will contain arbitrarily long runs of tiles in configuration A. Similarly, if $\alpha = 3/2 - \delta$, the row will contain arbitrarily long rows of tiles in configuration B.

Case $\alpha = 1$. Looking at the transition graph in Figure 3, we see that if $\alpha = 1$, a bottom sequence of $\cdots 111 \cdots$ can be obtain in two different ways. Call these configurations configuration A and configuration B. Notice that we can force the bottom labels of the row to contain arbitrarily long sequences of 1's separated by 0's by picking $\alpha = 1 - \delta$ for some small $0 < \delta$. Further, the only way this can happen is by alternating arbitrarily long occurrences of configuration A with arbitrarily long sequences of configuration B.

Similarly for cases $\alpha = 1/2$ and $\alpha = 1/3$, a small perturbation of α will produce arbitrarily long occurrences of each type of configuration. Since KC' contains all perturbations of angles of points in KC, KC' is dense in KC. ■

6 Explicit Return Time Bounds

We will now give explicit bounds on the on the size of the smallest rectangular configuration in KC that contains every $m \times n$ sub-configuration. The strategy will be to analyze the parameter space $[1/3, 2] \times \mathcal{T}$ to find intervals of parameters that have short $\hat{T}$-return times and then bound the $\hat{S}$-return times to such intervals. These return time bounds will then carry forward to (KC, T, S).

Definition. *Let $\mathcal{P}_{m,n}$ be the partition of $[1/3, 2] \times \mathcal{T}$ given by $m \times n$ configurations in KC. Specifically, $(\alpha, t) \sim (\alpha', t')$ if for $A = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\}$ we have*

$$K(\alpha, t, \mathbf{R}_{\lfloor \cdot \rfloor})|_A = K(\alpha', t', \mathbf{R}_{\lfloor \cdot \rfloor})|_A.$$

Definition. *For a partition $\mathcal{P}$ of $[1/3, 2] \times X$, let $\pi_\alpha(\mathcal{P})$ be the restriction of $\mathcal{P}$ to the fiber $\{\alpha\} \times X$.*

We will familiarize ourselves with the structure of $\mathcal{P}_{m,n}$. Let us consider $\mathcal{P}_{1,n}$. Putting the inverse-limit space $\mathcal{T}$ aside for a moment, let $\mathcal{P}_n$ be the partition of $[1/3, 2] \times [0, 1)$ such that $(\alpha, t) \sim (\alpha', t')$ if $(\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha, t))_0^{n-1} = (\mathbf{R}_{\lfloor\cdot\rfloor}(\alpha', t'))_0^{n-1}$.

After considering pre-images under rotation by an angle α, we see that $\pi_\alpha(\mathcal{P}_n)$ is precisely the partition generated by intervals whose endpoints are consecutive elements of $C_\alpha = \{0, -\alpha, -2\alpha, \ldots, -n\alpha \mod 1\}$. We view C_α as the places $[0, 1)$ needs to be "cut" to produce $\pi_\alpha(\mathcal{P}_n)$. Now, varying α, we see that $\mathcal{P}_n$ is produced by cutting $[1/3, 2] \times [0, 1)$ by the set of lines $L = \{(x, y) \in [1/3, 2] \times [0, 1) : y = -ix \mod 1 \text{ for some } i \leq n\}$.

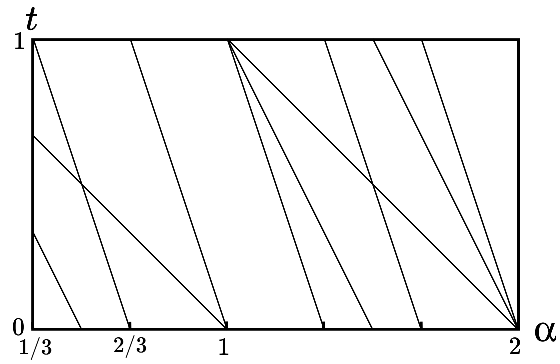


Figure 4: The partition $\mathcal{P}_3$.

Definition. *For $j \in \mathbb{N}$, define the σ-algebra $\mathscr{B}_j = (\mathrm{id} \times \mathrm{proj}_j)^{-1}(\mathscr{B})$ where $\mathscr{B}$ is the Borel σ-algebra defined on $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$.*

Informally, a partition $\mathcal{P}$ being $\mathscr{B}_j$-measurable means that for a point (α, t), α and $t \mod 6^j$ are all you need to determine the element of $\mathcal{P}$ in which it lies. Rephrased, $\mathcal{P}$ gives no extra information after the jth coordinate of $\mathcal{T}$. Consequently, $\mathcal{P}_{m,n}$ is $\mathscr{B}_j$-measurable for all $j \geq m$. Further, we may interchangeably talk about a $\mathscr{B}_j$-measurable partition of $[1/3, 2] \times \mathcal{T}$ and a Borel-measurable partition of $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$. Given a partition $\mathcal{A}$ of $[1/3, 2] \times \mathbb{R}/(6^j\mathbb{Z})$, we may define a partition $\mathcal{P}$ of $[1/3, 2] \times \mathcal{T}$ by the equivalence relation $(\alpha, t) \sim (\alpha', t')$ if $(\alpha, \mathrm{proj}_j(t))$ and $(\alpha', \mathrm{proj}_j(t'))$ lie in the same partition element of $\mathcal{A}$. In this case, we call $\mathcal{P}$ the *$\mathscr{B}_j$-measurable extension* of $\mathcal{A}$.

Let $A = [1/3, 2] \times [0, 6^i)$ which we will identify with $[1/3, 2] \times \mathbb{R}/(6^i\mathbb{Z})$. Given a finite collection, L, of lines in A, we form a partition of A up to a Lebesgue measure-zero set by taking the connected components of L^c. We call this the *geometric partition generated by* L.

Let's consider how $\mathcal{P}_{m,n}$ and our description of $\mathcal{P}_n$ arising from lines relate.

Definition. *Let $L^j_{a,\gamma} = \{(x, y) \in [1/3, 2] \times \mathbb{R}/(j\mathbb{Z}) : y = -ix + \gamma \mod j$ for some $0 \leq i \leq a\}$ be the set of lines with slopes in $\{0, -1, -2, \ldots, -a\}$ and offset γ.*

We can now view $\mathcal{P}_{1,n}$ as being the $\mathscr{B}_0$-measurable extension of the partition on $[1/3,2] \times [0,1)$ generated by $L^1_{n,0}$. Further, the boundary points of $\mathrm{id} \times \mathrm{proj}_j(\mathcal{P}_{1,n})$ are precisely the set $\bigcup_{i<6^j} L^{6^j}_{n,i}$.

Consider $\hat{f}^{-1}(\mathcal{P}_{1,n})$. Since $\hat{f}^{-1} : [1/3,2] \times \mathcal{T} \to [1/3,2] \times \mathcal{T}$ either multiplies by 3 or divides by 2 (and does so in each coordinate if we view $\hat{f}$ as acting on $(\mathbb{R}^2)^{\mathbb{N}}$), we see

$$\begin{aligned}\hat{f}|^{-1}_{[1/3,2/3]\times\mathbb{R}/(6^j\mathbb{Z})}\big(L^{6^j}_{n,i}\big) &= \\ &= \big\{(3x, 3y \bmod 6^j) : (x,y) \in L^{6^j}_{n,i} \cap ([1/3,2/3) \times \mathbb{R}/(6^j\mathbb{Z}))\big\} \\ &= L^{6^j}_{n,3i} \cap ([1,2] \times \mathbb{R}/(6^j\mathbb{Z})) \subset L^{6^j}_{n,3i}\end{aligned}$$

and

$$\begin{aligned}\hat{f}|^{-1}_{[2/3,2]\times\mathbb{R}/(6^{j+1}\mathbb{Z})}\big(L^{6^{j+1}}_{n,i}\big) &= \\ &= \big\{(\tfrac{x}{2}, \tfrac{y}{2} \bmod 6^j) : (x,y) \in L^{6^{j+1}}_{n,i} \cap ([2/3,2] \times \mathbb{R}/(6^{j+1}\mathbb{Z}))\big\} \\ &= L^{6^j}_{n,\frac{i}{2}} \cap ([1/3,1] \times \mathbb{R}/(6^{j+1}\mathbb{Z})) \subset L^{6^j}_{n,\frac{i}{2}}.\end{aligned}$$

Illustrated in Figure 5 is a truncation of $\mathrm{proj}_3(\hat{f}^{-i}\mathcal{P}_{1,1})$.

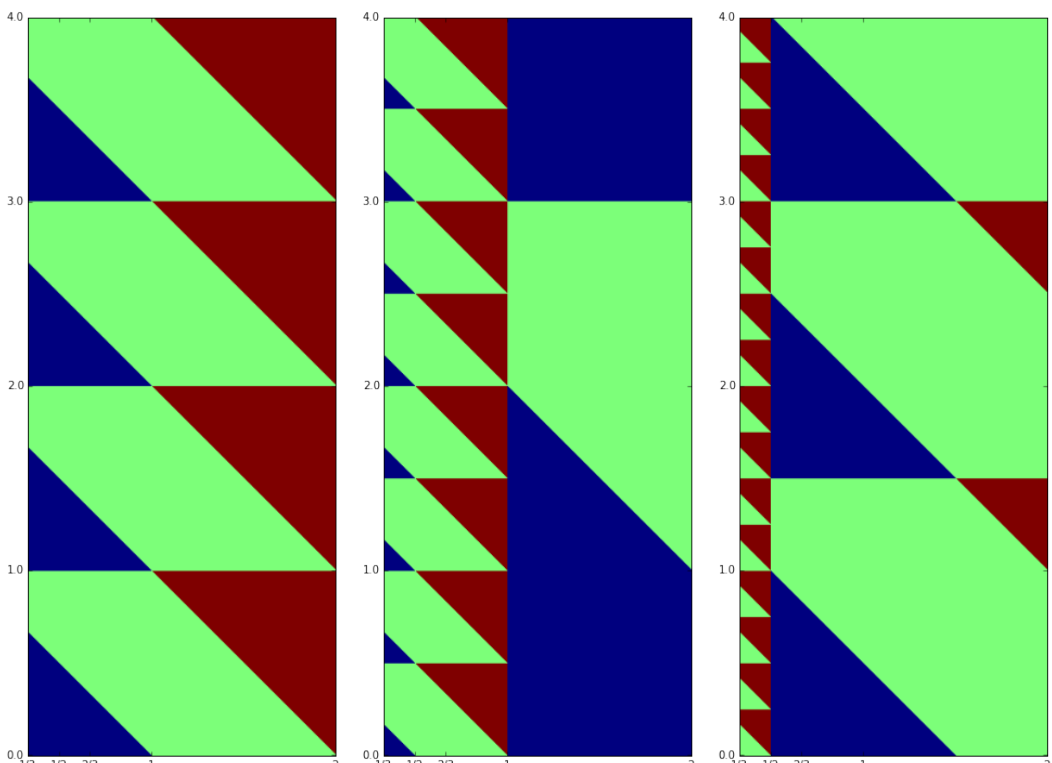


Figure 5: From left to right, the projection of $\mathcal{P}_{1,1}$, $\hat{f}^{-1}\mathcal{P}_{1,1}$, $\hat{f}^{-2}\mathcal{P}_{1,1}$ onto the third coordinate, truncated to lie in $[1/3,2] \times [0,4)$, and colored by whether the symbol at the zero position is 0, 1, or 2.

Definition. *Define* $\mathrm{rnd}_\alpha : \mathbb{R} \to \mathbb{Z}$ *by* $\mathrm{rnd}_\alpha(x) = n \in \mathbb{Z}$ *whenever* $x \in (n - \frac{\log 3}{\log 6} - \frac{\log \alpha}{\log 6}, n + \frac{\log 2}{\log 6} - \frac{\log \alpha}{\log 6})$.

Note that $(n - \frac{\log 3}{\log 6} - \frac{\log \alpha}{\log 6}, n + \frac{\log 2}{\log 6} - \frac{\log \alpha}{\log 6})$ is an interval of length 1, so rnd_α is defined everywhere but the countable set of endpoints. We will not attempt to use rnd_α in any situation where it is not defined.

Lemma 20. *Suppose* $\alpha \notin \mathbb{Q}$. *Then,* $\hat{f}^{-j}(\alpha, t) = (\frac{3^a}{2^b}\alpha, \frac{3^a}{2^b}t)$ *and* $f^{-j}(\alpha) = \frac{3^a}{2^b}\alpha$ *where*

$$a = j - b \approx \tfrac{\log 2}{\log 6} j \qquad \textit{and} \qquad b = \mathrm{rnd}_\alpha \left(\tfrac{\log 3}{\log 6} j \right) \approx \tfrac{\log 3}{\log 6} j.$$

The proof of Lemma 20 follows directly from solving the system $a+b = j$ and $\frac{3^a}{2^b}\alpha \in [1/3, 2]$ with the restriction that $a, b \in \mathbb{Z}$.

Proposition 21. *Let* B *be the boundary of* $\mathrm{id}\times\mathrm{proj}_m(\mathcal{P}_{m,n})$ *and let and* $b = \mathrm{rnd}_{f^{-m}(\alpha)}(m \log 3/\log 6)$. *Then* $\pi_\alpha(B)$ *is*

$$\pi_\alpha \left(\bigcup_{k<2^b 6^m} L^{6^m}_{n,\frac{k}{2^b}} \right).$$

Proof. Because the boundary of $\mathcal{P}_{m,n}$ is the boundary of $\bigvee_{i=0}^m f^{-i}(\mathcal{P}_{1,n})$, we see that $\pi_\alpha(L^{6^m}_{n,\frac{k}{2^b}})$ arises from applying $\hat{f}^{-m}$ to $\pi_{f^{-m}(\alpha)}(L^{6^{m+b}}_{n,k})$. This holds for every k, which completes the proof. ■

Iterating $\hat{f}^{-1}$ and observing how it moves the boundaries of partition elements motivates us to define the following refinement of $\mathcal{P}_{m,n}$.

Definition. *Let* $\mathcal{X}_{m,n}$ *be the* $\mathscr{B}_m$*-measurable extension of the partition generated by* L *where*

$$L = \bigcup_{k<2^m 6^m} L^{6^m}_{n,\frac{k}{2^m}}.$$

Proposition 22. $\mathcal{X}_{m,n}$ *is a refinement of* $\mathcal{P}_{m,n}$.

Proof. For a fixed α, let b_α be the b from Proposition 21, and notice that $b_\alpha \le m$. It now immediately follows that the set L defining $\mathcal{X}_{m,n}$ is a superset of the set of boundaries of $\mathrm{id} \times \mathrm{proj}_m(\mathcal{P}_{m,n})$, which completes the proof. ■

Definition. *For a partition* $\mathcal{P}$ *of* $\mathbb{R}$, *let* $\kappa\mathcal{P}$ *be the* coarseness *of* $\mathcal{P}$. *That is,*

$$\kappa\mathcal{P} = \inf_{I \in \mathcal{P}} \sup\{|b-a| : [a,b] \subset I\}.$$

If $\mathcal{P}$ *is a* $\mathscr{B}_j$*-measurable partition of* $\mathcal{T}$, *then* $\kappa\mathcal{P} = \kappa\mathrm{proj}_j(\mathcal{P})$.

Given $(\alpha, t) \in [1/3, 2] \times \mathcal{T}$, we would like to bound j such that $\mathcal{O}^j_{\hat{T}}(\alpha, t) = \{(\alpha, t), \hat{T}(\alpha, t), \ldots, \hat{T}^{j-1}(\alpha, t)\}$, the j-orbit of (α, t) under $\hat{T}$, intersects every partition element of $\pi_\alpha(\mathcal{P}_{m,n})$. We can address this in the following way.

Definition. *Let* $D^i_\ell(\alpha)$ *be the smallest* n *such that* $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha,t))$ *is* ℓ*-dense in* $\mathbb{R}/(6^i\mathbb{Z})$ *for any* t.

Note that the density of $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha,t))$ is equal to the density of $\mathrm{proj}_i(\mathcal{O}^n_{\hat{T}}(\alpha,t'))$, and so when computing $D^i_\ell(\alpha)$ we only need to consider a single t.

For a fixed α, we consider points in KC whose zeroth row has rotation number α. For a fixed t, consider the $n \times m$ configuration that arises based at $(0,0)$ corresponding to (α,t). We now see that for any $t' \in \mathcal{T}$, the maximum $\hat{T}$-waiting time to see an occurrence of this $n \times m$ configuration in the point corresponding to (α,t') is bounded by

$$D^m_{\kappa\pi_\alpha(\mathcal{P}_{m,n})}(\alpha) \le D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha).$$

Proposition 23. $\kappa\pi_\alpha(\mathcal{P}_{m,n}) \ge \kappa\pi_\alpha(\mathcal{X}_{m,n}) \ge \min\{|\alpha - \frac{p}{2^m q}| : q \le n\}$.

Proof. Since $\mathcal{X}_{m,n}$ is a refinement of $\mathcal{P}_{m,n}$, $\kappa\pi_\alpha(\mathcal{P}_{m,n}) \ge \kappa\pi_\alpha(\mathcal{X}_{m,n})$ follows trivially.

Let $\hat{\mathcal{X}}_{m,n}$ be the geometric partition generated by the lines $L = \bigcup_{k<2^m} L^1_{n,\frac{k}{2^m}}$. Recalling our description of $\mathcal{X}_{m,n}$ in terms of lines, we see that L corresponds exactly to the image under $\mathrm{id} \times \mathrm{proj}_0$ of the boundaries of the partition elements in $\mathcal{X}_{m,n}$. This shows that

$$\kappa\pi_\alpha(\hat{\mathcal{X}}_{m,n}) = \kappa\pi_\alpha(\mathcal{X}_{m,n}).$$

Thus, we will focus our attention on $\hat{\mathcal{X}}_{m,n}$.

Upon inspecting L, we see partition elements in $\pi_\alpha(\hat{\mathcal{X}}_{m,n})$ have endpoints in the set $E = \{\frac{k}{2^m}, -\alpha + \frac{k}{2^m}, \dots, -n\alpha + \frac{k}{2^m} \bmod 1 : k < 2^m\}$.

Fix α and observe $\kappa\pi_\alpha(\hat{\mathcal{X}}_{m,n}) = d$ is now given by the minimum distance between two points in E, which is

$$d = |-i\alpha + \tfrac{k}{2^m} - (-j\alpha + \tfrac{k'}{2^m}) + p| = |a\alpha + \tfrac{b}{2^m} + p|,$$

for appropriate $p, a, b \in \mathbb{Z}$. Since $\frac{d}{a} = |\alpha + \frac{b+p2^m}{a2^m}|$ for some $p \in \mathbb{Z}$, $a \le q$, and $-k < b < k$, the results follow immediately. ■

Having obtained a lower bound ℓ for $\kappa\pi_\alpha(\mathcal{P}_{m,n})$, we will now bound above the time it takes an orbit to become ℓ-dense.

Definition. *Define*

$$\mathcal{G}^{a,b} = \{\alpha : |\alpha - \tfrac{p}{q}| > \tfrac{1}{b} \textit{ for } q \le a \textit{ and } p, q \in \mathbb{N}\}.$$

Note that $\mathcal{G}^{a,b}$ could be empty, but a simple estimate shows that $\mathcal{G}^{a,b}$ is non-empty if $b > a^2$.

Proposition 24. $\alpha \in \mathcal{G}^{ka,b}$ *implies* $\{0, \alpha, 2\alpha, \dots, (kb-1)\alpha \bmod k\}$ *is* $\frac{1}{a}$*-dense in* $\mathbb{R}/(k\mathbb{Z})$.

Proof. For $x \in \mathbb{R}$, let $\|x\|_k$ represent the distance of x from $k\mathbb{Z}$. Suppose $\mathcal{G}^{ka,b} \ne \emptyset$, fix $\alpha \in \mathcal{G}^{ka,b}$, and let $q \in \mathbb{N}$ be the smallest number such that

$$\|q\alpha\|_k \le \tfrac{1}{a}.$$

Let $p \in \mathbb{Z}$ be such that $\|q\alpha\|_k = |q\alpha - kp|$. By the pigeonhole principle, $q \le ka$. By assumption, we have $|\alpha - \frac{kp}{q}| > \frac{1}{b}$ and so

$$\|q\alpha\|_k = |q\alpha - kp| > \tfrac{q}{b}.$$

It would suffice to show that the points

$$\begin{aligned} X &= \{0, \|q\alpha\|_k, 2\|q\alpha\|_k, \dots, (\lceil b/q \rceil k - 1)\|q\alpha\|_k\} \\ &= \{0, q\alpha, 2q\alpha, \dots, (\lceil b/q \rceil k - 1)q\alpha \bmod k\} \end{aligned}$$

are $\frac{1}{a}$-dense since they form a subset of $\{0, \alpha, 2\alpha, \dots, (kb-1)\alpha \bmod k\}$. Since consecutive points in X are separated by a distance of less that $\frac{1}{a}$, we need only show that the last point satisfies $(\lceil b/q \rceil k - 1)\|q\alpha\|_k \ge k - 1/a$. But this is implied by the fact that $\frac{b}{q}\|q\alpha\|_k > 1$, which completes the proof. ■

Proposition 25. *$\alpha \in \mathcal{G}^{ka,b}$ implies $\{0, \alpha, 2\alpha, \dots, ka\alpha \bmod k\}$ is $\frac{1}{b}$-sparse in $\mathbb{R}/(k\mathbb{Z})$. That is, no two points are within $1/b$ of each other.*

Proof. Suppose $\mathcal{G}^{ka,b} \neq \emptyset$. Fix $\alpha \in \mathcal{G}^{ka,b}$ and note that to prove $\frac{1}{b}$-sparsity of $\{0, \alpha, 2\alpha, \dots, ka\alpha \bmod k\}$ we only need to show $\|r\alpha\|_k > \frac{1}{b}$ for all $0 < r \le ka$.

Choose p, q to minimize $|q\alpha - kp|$ subject to $0 < q \le ka$. We then have that $\|r\alpha\|_k$ is minimized by

$$\|q\alpha\|_k = |q\alpha - kp| \ge |\alpha - \tfrac{kp}{q}| > \tfrac{1}{b},$$

with the last inequality following by assumption. ■

The previous propositions show a symmetry in $\mathcal{G}^{a,b}$. Namely, if $\alpha \in \mathcal{G}^{a,b}$, then the b-orbit of rotation by α is $1/a$-dense and the $(a+1)$-orbit of rotation by α is $1/b$-sparse.

Corollary 26. *If $\alpha \in \mathcal{G}^{2^m n,b}$ then $\kappa\pi_\alpha(\mathcal{X}_{m,n}) > \frac{1}{b}$.*

Proof. Fix $\alpha \in \mathcal{G}^{2^m n,b}$. By Proposition 23, $\kappa\pi_\alpha(\mathcal{X}_{m,n}) \ge \min\{|\alpha - \frac{p}{2^m q}| : q \le n \text{ and } p, q \in \mathbb{N}\}$. By the assumption that $\alpha \in \mathcal{G}^{2^m n,b}$, we have $|\alpha - \frac{p}{2^m q}| > \frac{1}{b}$. ■

Proposition 27. *If $\alpha \in \mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$ then*

$$D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha) \le 6^m c.$$

Proof. Fix $\alpha \in \mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$. Since $\alpha \in \mathcal{G}^{2^m n,b}$, Corollary 26 implies $\kappa\pi_\alpha(\mathcal{X}_{m,n}) > \frac{1}{b}$ and so $\kappa\pi_\alpha(\mathrm{proj}_m(\mathcal{X}_{m,n})) > \frac{1}{b}$. By Proposition 24 applied to $\mathcal{G}^{6^m b,c}$, we have that $E = \{0, \alpha, \dots, (6^m c - 1)\alpha \bmod 6^m\}$ is $\frac{1}{b}$-dense in $[0, 6^m]$, and so E intersects every partition element of $\pi_\alpha(\mathcal{X}_{m,n})$, which completes the proof. ■

We have identified α's that give us good return times, but $\mathcal{G}^{2^m n,b} \cap \mathcal{G}^{6^m b,c}$ could be empty. Next we will find constraints on b, c to avoid this and guarantee us some useful properties.

Definition. *Given a set X and a collection of sets $\mathcal{C}$, we say $(X, \mathcal{C})$ has the* intersection property *if for all $I \in \mathcal{C}$, $X \cap I \neq \emptyset$. If $X \subset \mathbb{R}$, we say X is δ-*fat *relative to $\mathcal{C}$ if for all $I \in \mathcal{C}$, $X \cap I$ contains an interval of width δ.*

Definition. *Let $\mathcal{F}_n$ be the partition of $\mathbb{R}$ whose elements are of the form $[a,b)$ where a,b are consecutive points in $\{\frac{p}{q} : q \leq n\}$. That is $\mathcal{F}_n$ is the partition of $\mathbb{R}$ into half-open intervals whose endpoints are consecutive Farey fractions with denominator bounded by n.*

Proposition 28. *Let $p : [1/3,2] \times \mathcal{T} \to [1/3,2]$ be projection onto the first coordinate. Let $X \subset \mathbb{R}$. If $(X, \mathcal{F}_{2^m n})$ has the intersection property, then for any element $E \in \mathcal{X}_{m,n}$, $X \cap p(E) \neq \emptyset$.*

Proof. Let $\hat{\mathcal{X}}_{m,n} = \mathrm{id} \times \mathrm{proj}_0(\mathcal{X}_{m,n})$ and note it is sufficient to show that if $(X, \mathcal{F}_{2^m n})$ has the intersection property, then for any element $E \in \hat{\mathcal{X}}_{m,n}$, we have $X \cap p(E) \neq \emptyset$.

Recalling the description of $\hat{\mathcal{X}}_{m,n}$ in terms of lines, we see that $\hat{\mathcal{X}}_{m,n}$ consists of polygonal regions whose corners have coordinates of the form $\frac{p}{2^m q}$ for some $q \leq n$. Since every element of $\hat{\mathcal{X}}_{m,n}$ contains an open set, we see that for all $P \in \hat{\mathcal{X}}_{m,n}$, there exists $I \in \mathcal{F}_{2^m n}$ so that $I \subset p(P)$ (possibly ignoring some points along the boundary of P), which completes the proof. ■

Proposition 29. *If $b \geq 4a^2$, $c^2 \geq 4b$, and $d \geq 4c^2$ then $\mathcal{G}^{a,b} \cap \mathcal{G}^{c,d}$ is $\frac{2}{d}$-fat relative to $\mathcal{F}_a$.*

Proof. By definition $\mathcal{G}^{x,y}$ is constructed by removing balls of radius $1/y$ centered at points $\frac{p}{q}$ with $q \leq x$. If $q, q' \leq x$, then $|\frac{p}{q} - \frac{p'}{q'}| > \frac{1}{x^2}$. Thus, if $y > 4x^2$, not only will $\mathcal{G}^{x,y}$ intersect every element of $\mathcal{F}_x$, but it will do so with diameter at least

$$\frac{1}{x^2} - \frac{2}{y} = \frac{1}{2x^2}.$$

Suppose a,b,c,d satisfy $b \geq 4a^2$, $c^2 \geq 4b$, and $d \geq 4c^2$. Every gap in $\mathcal{G}^{c,d}$ is of size $\frac{2}{d} < \frac{1}{2c^2}$ and every interval in $\mathcal{G}^{c,d}$ has size at least $\frac{1}{2c^2}$. Thus, the intersection of $\mathcal{G}^{c,d}$ with an interval of width $\frac{1}{2b}$ must contain an interval of width at least

$$\min\left\{\frac{1}{2c^2}, \frac{1}{2b} - \frac{2}{2c^2}\right\} \geq \min\left\{\frac{1}{2c^2}, \frac{2}{c^2} - \frac{1}{c^2}\right\} = \frac{1}{2c^2} \geq \frac{2}{d}.$$

Noticing that the smallest interval in $\mathcal{G}^{a,b}$ is of size at least $\frac{2}{b} > \frac{1}{2b}$ completes the proof. ■

We can now identify a set of α's that have good waiting times.

Definition. *Let $\mathcal{W}_{n\times m} = \mathcal{G}^{a,b} \cap \mathcal{G}^{c,d}$ where $a = 2^m n$, $b = 2^{2m+2}n^2$, $c = 6^m 2^{2m+2} n^2$, and $d = 6^{4m+3}n^4$.*

Notice that the parameters a,b,c,d in $\mathcal{W}_{n\times m}$ were carefully chosen to satisfy the conditions of Proposition 29 and Proposition 27.

Theorem 30. *Let c be an $n \times m$ configuration in KC and $A = \{0,\ldots,m-1\} \times \{0,\ldots,n-1\}$. Then there exists an interval $I_c \subset \mathcal{W}_{n\times m}$ of width $2/(6^{4m+3}n^4)$ so that for every $\alpha \in I_c$ and every $t \in \mathcal{T}$, there exists a $j < 6^{5m+3}n^4$ so that*

$$K \circ \hat{T}^j(\alpha, t)|_A = c.$$

Proof. Given the framework we have established, the proof is straightforward.

Proposition 29 tells us that $\mathcal{W}_{n\times m}$ is $2/(6^{4m+3}n^4)$-fat relative to $\mathcal{F}_{2^m n}$, and so by Proposition 28, we have that there exists an interval $I_c \subset \mathcal{W}_{n\times m}$ of width $2/(6^{4m+3}n^4)$ so that for every $\alpha \in I_c$ there exists $t \in \mathcal{T}$ so $K(\alpha, t)|_A = c$.

Fix I_c and $\alpha \in I_c$. By Proposition 27,

$$D^m_{\kappa\pi_\alpha(\mathcal{X}_{m,n})}(\alpha) \le 6^m 6^{4m+3} n^4 = 6^{5m+3} n^4,$$

and so we will see c in less than $6^{5m+3}n^4$ applications of $\hat{T}$, which completes the proof. ■

Theorem 30 gives the bulk of the proof of Theorem C. If we have an $n \times m$ configuration c in mind, we know there is an open interval I_c of angle parameters where we will see c in a horizontal orbit of no more than $6^{5m+3}n^4$ steps. Since orbits under $\hat{S}$ are dense in the first coordinate, we know that if we bound how long it takes for an $\hat{S}$-orbit (equivalently an f-orbit) to become $|I_c|$-dense, we have a bound on the minimum size of a rectangle that contains the configuration c.

6.1 Asymptotic Density of Orbits Under f

Definition (Irrationality Measure)**.** *For a number $\alpha \in \mathbb{R}$, the* irrationality measure *of α is*

$$\eta(\alpha) = \inf\left\{\gamma : \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^\gamma} \text{ for only finitely many } p, q \in \mathbb{Z}\right\}.$$

Proposition 31 (Rhin [7])**.** *For $u_0, u_1, u_2 \in \mathbb{Z}$ and $H = \max\{|u_1|, |u_2|\}$, we have that if H is sufficiently large,*

$$|u_0 + u_1 \log 2 + u_2 \log 3| \ge H^{-7.616},$$

and if $H \ge 2$, we have the universal bound

$$|u_0 + u_1 \log 2 + u_2 \log 3| \ge H^{-13.3}.$$

Corollary 32. *$\eta(\log 2/\log 6) \le 8.616$ and $\left|\frac{\log 2}{\log 6} - \frac{p}{q}\right| \ge \frac{1/\log 6}{q^{14.3}}$ if $q \ge 2$.*

Proof. Let $x = \left|\frac{\log 2}{\log 6} - \frac{p}{q}\right|$. By algebraic manipulation, we deduce

$$xq \log 6 = |(q-p)\log 2 - p \log 3|.$$

And so by Proposition 31 and the fact that $\max\{|q-p|, |p|\} \le q$, we have that asymptotically, $xq\log 6 \ge q^{-7.616}$, which produces a bound of $x \ge \frac{1/\log 6}{q^{8.616}}$. Alternatively, we may use the bound $xq \log 6 \ge q^{-13.3}$, which holds for all $q \ge 2$. ■

Proposition 33. *Fix $\delta > 0$ and let $k_\ell \ge (\frac{3}{\ell \log 6})^{8.616+\delta}$. Then, for sufficiently small ℓ, the k_ℓ-orbit of any $x \in [1/3, 2]$ under f is ℓ-dense. That is*

$$\{x, f(x), f^2(x), \dots f^{k_\ell - 1}(x)\}$$

is ℓ-dense for any $x \in [1/3, 2]$. Further, if $k_\ell \ge (\frac{3}{\ell \log 6})^{14.3} \log 6$ and $1/\ell \ge 2$, then the k_ℓ orbit of any point $x \in [1/3, 2]$ under f is ℓ-dense.

Proof. Let ϕ be the conjugacy from Proposition 7 between f and rotation by $\frac{\log 2}{\log 6}$. We have that $|\phi'|$ attains a maximum value of $\frac{3}{\log 6}$. Thus, to ensure an orbit segment under f is ℓ-dense, we must have that the image of an orbit segment under $\phi \circ f \circ \phi^{-1} = R_{\frac{\log 2}{\log 6}}$ is $\frac{\ell \log 6}{3}$-dense. Let $\eta = \eta(\log 2/\log 6)$ be the irrationality measure of $\log 2/\log 6$. Fix $\delta > 0$. We then have, by the definition of the irrationality measure, $\log 2/\log 6 \in \mathcal{G}^{k,k^{\eta+\delta}}$ for all sufficiently large k. Applying Proposition 24 and using Corollary 32 to bound η now completes the proof of the first claim.

For the second claim, note that Corollary 32 implies that $\frac{\log 2}{\log 6} \in \mathcal{G}^{k,k^{14.3}\log 6}$ for any $k \geq 2$. The proof then follows similarly. ■

Theorem (Theorem C)**.** *Let $\eta = \eta(\log 2/\log 6)$. Every legal $n \times m$ configuration in KC occurs in every $B \times A$ configuration in KC where*

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{\eta} < 6^{34.464m+25} n^{34.464} \qquad \textit{and} \qquad B = 6^{5m+3} n^4$$

for sufficiently large $m + n$.

Further, for all m, n we have that a copy of every legal $n \times m$ configuration in KC occurs in every $B \times A$ configuration in KC where

$$A = \left(\frac{324}{\log 6} 6^{4m} n^4\right)^{14.3} \log 6 \qquad \textit{and} \qquad B = 6^{5m+3} n^4$$

Proof. Let $C = \{0, \ldots, n-1\} \times \{0, \ldots, m-1\}$ and let c be a legal $n \times m$ configuration. Fix $I_c \subset \mathcal{W}_{n\times m}$ as in Theorem 30. We now have that for any $(\alpha, t) \in I_c \times \mathcal{T}$, $K \circ \hat{T}^j(\alpha, t)|_C = c$ for some $j < 6^{5m+3} n^4$.

Since I_c is of length at least $2/(6^{4m+3} n^4)$, by Proposition 33 with $\ell = 2/(6^{4m+3} n^4)$, we see that for any $(\alpha, t) \in [1/3, 2] \times \mathcal{T}$, we have $\hat{S}^j(\alpha, t) \in I_c \times \mathcal{T}$ for some $j < (3 \cdot 6^{4m+3} n^4/(2 \log 6))^{\eta}$.

We now have a bound on how many applications of $\hat{T}$ and $\hat{S}$ it takes to land in a particular element of $\mathcal{P}_{n,m}$, which gives bounds on A and B.

Alternatively, using the second part of proposition 33 with $\ell = 2/(6^{4m+3} n^4)$ we get a bound for all $m \geq 2$. ■

Acknowledgements. I would like to thank the referee for a careful reading and useful suggestions.

References

[1] C. A. Berenstein and D. Lavine. On the number of digital straight line segments. *Pattern Analysis and Machine Intelligence, IEEE Transactions on*, 10(6):880–887, Nov 1988.

[2] B. Durand, G. Gamard, and A. Grandjean. Aperiodic tilings and entropy. In *18th International Conference on Developments in Language Theory, DLT 2014, Ekaterinburg, Russia*, 2014.

[3] S. Eigen, J. Navarro, and V. S. Prasad. An aperiodic tiling using a dynamical system and Beatty sequences. In *Dynamics, ergodic theory, and geometry*, volume 54 of *Math. Sci. Res. Inst. Publ.*, pages 223–241. Cambridge Univ. Press, Cambridge, 2007.

[4] N. P. Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[5] I. Liousse. PL homeomorphisms of the circle which are piecewise C^1 conjugate to irrational rotations. *Bull. Braz. Math. Soc. (N.S.)*, 35(2):269–280, 2004.

[6] M. Lothaire. *Algebraic combinatorics on words*, volume 90 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2002. A collective work by Jean Berstel, Dominique Perrin, Patrice Seebold, Julien Cassaigne, Aldo De Luca, Steffano Varricchio, Alain Lascoux, Bernard Leclerc, Jean-Yves Thibon, Veronique Bruyere, Christiane Frougny, Filippo Mignosi, Antonio Restivo, Christophe Reutenauer, Dominique Foata, Guo-Niu Han, Jacques Desarmenien, Volker Diekert, Tero Harju, Juhani Karhumaki and Wojciech Plandowski, With a preface by Berstel and Perrin.

[7] G. Rhin. Approximants de Padé et mesures effectives d'irrationalité. In *Séminaire de Théorie des Nombres, Paris 1985–86*, volume 71 of *Progr. Math.*, pages 155–164. Birkhäuser Boston, Boston, MA, 1987.

[8] E. A. Robinson, Jr. The tilings of Kari and Culik. Presented at Numeration: Mathematics and Computer Science, CIRM, Marseilles, 2009.