Statics and Dynamics of Magnetic Vortices

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Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations:

$$-\Delta_A \psi = \kappa^2(1 - |\psi|^2)\psi$$
$$curl^2 A = Im(\bar{\psi} \nabla_A \psi)$$

where $(\psi, A) : \mathbb{R}^2 \to \mathbb{C} \times \mathbb{R}^2$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

Origin of Ginzburg-Landau Equations

Superconductivity. $\psi : \mathbb{R}^2 \to \mathbb{C}$ is called the *order parameter*; $|\psi|^2$ gives the density of (Cooper pairs of) superconducting electrons.

$A : \mathbb{R}^2 \to \mathbb{R}^2$ is the magnetic potential. The r.h.s. of the equation for A is the superconducting current.

Particle physics. ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively.
(One can think of A as a connection on the principal $U(1)$- bundle $\mathbf{R}^2 \times U(1)$, and ψ, as the section of this bundle.)

Similar equations appear in the theory of superfluidity and of fractional quantum Hall effect.

Ginzburg-Landau Energy

Ginzburg-Landau equations are the Euler-Lagrange equations for the *Ginzburg-Landau energy functional*

$$\mathcal{E}_\Omega(\psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \psi|^2 + (\mathrm{curl}\, A)^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \right\}.$$

Superconductors: $\mathcal{E}(\psi, A)$ is the difference in (Helmhotz) free energy between the superconducting and normal states.

In the $U(1)$ Higgs model case, $\mathcal{E}_\Omega(\psi, A)$ is the energy of a static configuration in the $U(1)$ Yang-Mills-Higgs classical gauge theory.

Symmetries

The gauge symmetry: for any regular $\eta : \mathbb{R}^2 \to \mathbb{R}$,

$$\psi \mapsto e^{i\eta}\psi, \qquad A \mapsto A + \nabla\eta;$$

Translation symmetry: for each $t \in \mathbb{R}^2$,

$$\psi(x) \mapsto \psi(x+t), \qquad A(x) \mapsto A(x+t).$$

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Rotation and reflection symmetry: for each $R \in O(2)$

$$\psi(x) \mapsto \psi(Rx), \qquad A(x) \mapsto R^{-1}A(Rx).$$

Quantization of Flux

Finite energy states (ψ, A) are classified by the topological degree

$$deg(\psi) := deg\left(\frac{\psi}{|\psi|}\bigg|_{|x|=R}\right),$$

where $R \gg 1$. For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi deg(\psi) \in 2\pi\mathbb{Z},$$

where $B := curlA$ is the magnetic field associated with the vector potential A.

Vortices

Besides the homogenous solutions ($\psi \equiv 1,\ A \equiv 0$) (perfect superconductor) and ($\psi = 0,\ \text{curl}\, A = \text{constant}$) (normal metal), the Ginzburg-Landau equations have "radially symmetric" (more precisely *equivariant*) solutions of the form

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a_n(r)\nabla(n\theta) \ ,$$

where n is an integer and (r, θ) are the polar coordinates of $x \in \mathbb{R}^2$. Note: $deg(\psi^{(n)}) = n$.

The pair $(\psi^{(n)}, A^{(n)})$ is called the *n-vortex* (*magnetic* or *Abrikosov* in the case of superconductors, and *Nielsen-Olesen* or *Nambu string* in the particle physics case).

Type I and II Superconductors

There is the critical value $\kappa = 1/\sqrt{2}$, that separates superconductors into two classes with different properties:

$\kappa < 1/\sqrt{2}$: Type I superconductors, exhibit first-order phase transitions from the non-superconducting state to the superconducting state (essentially, all pure metals);

$\kappa > 1/\sqrt{2}$: Type II superconductors, exhibit second-order phase transitions and the formation of vortex lattices (dirty metals and alloys).

For $\kappa = 1/\sqrt{2}$, Bogomolnyi has shown that the Ginzburg-Landau equations are equivalent to a pair of first-order equations. Using this Taubes described completely solutions of a given degree.

Stability/Instability of Vortices

Theorem

1. *For Type I superconductors all vortices are stable.*
2. *For Type II superconductors, the ± 1-vortices are stable, while the n-vortices with $|n| \geq 2$, are not.*

The statement of Theorem I was conjectured by Jaffe and Taubes on the basis of numerical observations (Jacobs and Rebbi, ...).

Abrikosov Lattice States

Consider states (ψ, A), defined on all of $\mathbb{R}^2$, but such that physical quantities, $|\psi|^2$ and $\text{Im}(\bar{\psi}\nabla_A\psi)$, are doubly-periodic with respect to some lattice $\mathcal{L}$. By the gauge invariance for such (ψ, A),

$$\forall t \in \mathcal{L}, \ \exists g_t : \ \psi(x+t) = e^{ig_t(x)}\psi(x), \ A(x+t) = A(x) + \nabla g_t(x).$$

Such states will be called ($\mathcal{L}$-) *gauge or Abrikosov lattice states*.

Critical Magnetic Fields

There are two key critical magnetic fields:
H_{c1} is the field at which the first vortex enters the superconducting sample.

H_{c2} is the field at which the normal material becomes superconducting.

Type I superconductors: $H_{c1} > H_{c2}$;
Type II superconductors: $H_{c1} < H_{c2}$.

Consider Type II superconductors in two regimes: either

$$0 < H_{c2} - H \ll H_{c2},$$

or

$$0 < H - H_{c1} \ll H_{c1}.$$

Abrikosov Lattices

Let $b := \langle B \rangle_\Omega$ be the average magnetic flux per basic lattice cell Ω.

Theorem

Let $\kappa > 1/2$. For every $\mathcal{L}$ and every b, $0 < H_{c2} - b \ll H_{c2}$,

(1) There exist non-trivial $\mathcal{L}$-lattice solution with one quantum of flux per cell and with average magnetic flux per cell equal to b;

(2) The minimum of the average energy per cell is achieved on the solution corresponding to the triangular lattice.

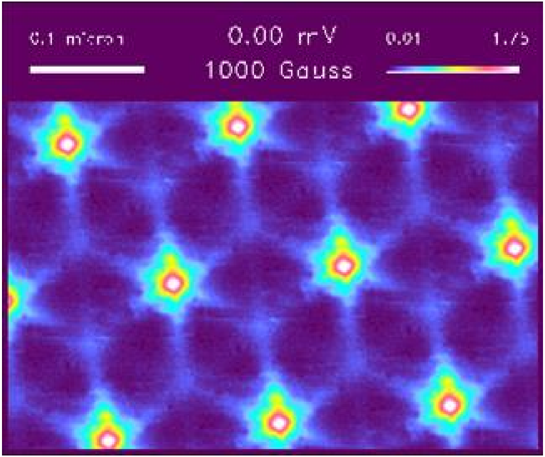
Theorem

The solution above is linearly stable.

Theorem

Existence and stability for $0 < H - H_{c1} \ll H_{c1}$.

Abrikosov Lattice. Experiment



Time-Dependent Eqns. Superconductivity

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\phi)\psi = \Delta_A\psi + \kappa^2(1 - |\psi|^2)\psi$$
$$\sigma(\partial_t A - \nabla\phi) = -curl^2 A + Im(\bar{\psi}\nabla_A\psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \text{Im}(\overline{\psi}\nabla_A\psi)$, and $J_n = \sigma E$.

Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations

$$\partial_t^2\psi = \Delta_A\psi + \kappa^2(1-|\psi|^2)\psi$$
$$\partial_t^2 A = -curl^2 A + Im(\bar{\psi}\nabla_A\psi),$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics (written here in the *temporal gauge*).

Linear Stability

A solution (ψ, A) is *linearly stability* if the Hessian $\mathcal{E}''_\Omega(\psi, A)$ satisfies

$$\text{null}\, \mathcal{E}''_\Omega(\psi, A) = \mathcal{Z},$$

$$\langle v, \mathcal{E}''_\Omega(\psi, A)v\rangle > 0, \ \forall v \perp \mathcal{Z}.$$

Here

$$\mathcal{Z} = \{G_\gamma : \ g \in H_2(\mathbb{R}^2, \mathbb{R})\},$$

the space of gauge symmetry zero modes,

$$G_\gamma := (i\gamma\psi, \nabla\gamma).$$

Properties of Hessian

- The Hessian $\mathcal{E}''_{\Omega}(\psi, A)$ is a real-linear operator;
- is symmetric ($\langle w', Lw\rangle = \langle Lw', w\rangle$) in the inner product

$$\langle w, w'\rangle = \int \mathrm{Re}\,\overline{\xi}\xi' + \alpha\cdot\alpha',$$

  where $w = (\xi, \alpha)$, etc;
- has gauge symmetry zero modes, $G_\gamma := (i\gamma\psi, \nabla\gamma)$.

We extend $\mathcal{E}''_{\Omega}(\psi, A)$ to a complex-linear operator L (also called Hessian) and study this operator.

Idea of Proof of Stability

The key point: The Hessian $L \sim \mathcal{E}''_\Omega(\psi, A)$ commutes with magnetic translations:

$$T_t L = L T_t.$$

Here the magnetic translation T_t are given by

$$T_t = e^{-i\frac{b}{2}t\wedge x} S_t,$$

where S_t is the translation operator $S_t f(x) = f(x+t)$ and

$$t \wedge x = t_1 x_2 - t_2 x_1.$$

Magnetic translational symmetry

Recall T_t is the magnetic translation defined by

$$T_t = e^{-i\frac{b}{2}t\wedge x} S_t, \qquad S_t f(x) = f(x+t).$$

The particular form of the magnetic translation is due to our choice of gauge.

Using the flux quantization relation $bt \wedge s = 2\pi n$, one can show:

$$T_{t+s} = e^{-i\frac{b}{2}t\wedge s} T_t T_s.$$

Hence T_t defines a unitary projective group representation of $\mathcal{L}$ on $L^2(\mathbb{R}^2; \mathbb{C}) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$. It can be lifted to a standard representation τ_t.

Direct Fibre Integral (Bloch Decomposition)

We use $\tau_t L = L\tau_t$, and the representation τ_t to decompose the operator L into the fiber direct integral

$$ULU^{-1} = \int_{\hat{\Omega}}^{\oplus} L_k d\mu_k$$

on the space

$$\mathscr{H} = \int_{\hat{\Omega}}^{\oplus} \mathscr{H}_k d\mu_k,$$

where $\hat{\Omega}$ is the fundamental cell of the reciprocal lattice (the dual group to $\mathcal{L}$ under addition mod the reciprocal lattice), and $d\mu_k = \frac{dk}{|\hat{\Omega}|}$ is the Haar measure on $\hat{\Omega}$.

Fibre Spaces and Operators

Above, $U : L^2(\mathbb{R}^2;\mathbb{C}) \times L^2(\mathbb{R}^2;\mathbb{R}^2) \to \mathscr{H}$ is a unitary operator given by

$$(Uv)_k(x) = \sum_{t\in\mathcal{L}} \chi_t^{-1}\tau_t v(x),$$

where $\chi : \mathcal{L} \to U(1)$ is a character of the representation τ_t of $\mathcal{L}$, explicitly given by

$$\chi_t = e^{ik\cdot t}, \quad k \in \hat{\Omega},$$

L_k is the restriction of the operator L to $H_2(\Omega)$, satisfying

$$\tau_t v(x) = \chi_t v(x), t \in \text{basis}.$$

Θ-function

In the leading order the analysis of the ground states of the fiber operators is reduced to construction of an entire function $\Theta(z),\ z = x^1 + ix^2 \in \mathbb{C}$, satisfying the periodicity conditions

$$\Theta(z+1) = \Theta(z),$$

$$\Theta(z+\tau) = e^{i(k_2 - k_1\tau)} e^{-2inz} e^{-in\tau z} \Theta(z),$$

where $\tau = \frac{r'}{r}$, with r, r' a basis for a lattice $\mathcal{L} \subseteq \mathbb{C}$.
(The complex number τ characterizes (the shape of) the lattice $\mathcal{L}$. Θ has inherited these conditions from the gauge-periodicity of ψ.)

Lowest Eigenfunctions (Leading Order)

The first relation above ensures that Θ have an absolutely convergent Fourier expansion

$$\Theta(z) = \sum_{m=-\infty}^{\infty} c_m e^{2miz}.$$

The second relation leads to the relation for the coefficients:

$$c_{m+n} = e^{i(k_1\tau - k_2)} e^{in\pi\tau} e^{2mi\pi\tau} c_m.$$

$\rightarrow$ Such functions are parameterized by $c_0, \ldots, c_{n-1}$.
This gives the ground states of the fibers, L_k, of linearized operator L in the leading order. A perturbation theory finishes the proof.

Consider a dynamical problem with initial conditions, describing several vortices, with the centers at points z_1, $z_2, \ldots$ and with the degrees n_1, n_2, ... , glued together, e.g.

$$\psi_{\underline{z},\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j),$$

$$A_{\underline{z},\chi}(x) = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x) \ ,$$

where $\underline{z} = (z_1, z_2, \ldots)$ and χ is an arbitrary real function.
We will assume that $R(\underline{z}) := \min_{j \neq k} |z_j - z_k| \gg 1$.

The *superconductor model*: for initial data (ψ_0, A_0) close to some $(\psi_{\underline{z}_0,\chi_0}, A_{\underline{z}_0,\chi_0})$ with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} \le \epsilon \ll 1$ we have

$$(\psi(t), A(t)) = (\psi_{\underline{z}(t),\chi(t)}, A_{\underline{z}(t),\chi(t)}) + O(\epsilon \log^{1/4}(1/\epsilon))$$

and that the vortex dynamics is governed by the system

$$\gamma_{n_j}\dot{z}_j = -\nabla_{z_j} W(\underline{z}) + O(\epsilon^2 \log^{3/4}(1/\epsilon)).$$

Here $W(\underline{z}) \sim \sum_{j\neq k}(const)n_j n_k \frac{e^{-|z_j-z_k|}}{\sqrt{|z_j-z_k|}}$ is the *effective energy* and $\gamma_n > 0$.

Vortex Dynamics: $U(1)$-Higgs Model

The *Higgs model*: for times up to $O\left(\frac{1}{\sqrt{\epsilon}}\log\left(\frac{1}{\epsilon}\right)\right)$, the effective dynamics is given by

$$\gamma_{n_j}\ddot{z}_j = -\nabla_{z_j}W(\underline{z}(t)) + o(\epsilon).$$

with $\gamma_n > 0$ and with the same effective energy/Hamiltonian

$$W(\underline{z}) \sim \sum_{j\neq k}(const)n_j n_k \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}}.$$

Open Problems

- Extend the previous results to the Chern-Simons equations, appearing in the macroscopic theory of the fractional quantum Hall effect (FQHE);
- Nonlinear stability of Abrikosov lattices;
- Abrikosov lattices at higher fluxes;
- Asymptotic dynamics of vortices.

Thank-you for your attention.