Special cycles and derivatives of Eisenstein series

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"A man hears what he wants to hear
and disregards the rest."
Simon and Garfunkel
The boxer

This article is an expanded version of a lecture given at the conference on Special Values of Rankin L-Series at MSRI in December of 2001. I have tried to retain some of the tone of an informal lecture. In particular, I have attempted to outline, in very broad terms, a program involving relations among:

(i) algebraic cycles
(ii) Eisenstein series and their derivatives
(iii) special values of Rankin-Selberg L-functions and their derivatives,

ignoring many important details and serious technical problems in the process. I apologize at the outset for the very speculative nature of the picture given here. I hope that, in spite of many imprecisions, the sketch will provide a context for a variety of particular cases where precise results have been obtained. Recent results on one of these, part of an ongoing joint project with Michael Rapoport and Tonghai Yang on which much of the conjectural picture is based, are described in Yang's article in this volume, [**81**]. A less speculative discussion of some of this material can be found in [**43**], [**45**], and [**46**].

I would like to thank my collaborators B. Gross, M. Harris, J. Millson, S. Rallis, M. Rapoport and T. Yang for generously sharing their mathematical ideas and for their support over many years. I would also like to thank R. Borcherds, J.-B. Bost, J. Cogdell, J. Funke, R. Howe, D. Kazhdan, K. Keating, J. Kramer, U. Kühn, J.S. Li, J. Schwermer, and D. Zagier for helpful discussions, comments and suggestions. Finally, I would like to thank Henri Darmon and Shou-Wu Zhang for organizing such an enjoyable and inspiring program and for their very helpful suggestions which significantly improved this article. Finally, I would like to thank MSRI for its ever excellent hospitality.

[1] Partially supported by NSF grant DMS-9970506 and by a Max-Planck Research Prize from the Max-Planck Society and Alexander von Humboldt Stiftung.

I
An attractive family of varieties

§1. Shimura varieties of orthogonal type.

We begin with the following data:[2]

$$
\begin{aligned}
V,\ (\ ,\ ) &= \text{inner product space over } \mathbb{Q}\\
\mathrm{sig}(V) &= (n,2)\\
G &= \mathrm{GSpin}(V) \qquad (1.1)\\
D &= \{\ w\in V(\mathbb{C}) \mid (w,w)=0,\ (w,\bar{w})<0\ \}/\mathbb{C}^\times \ \subset\ \mathbb{P}(V(\mathbb{C}))\\
n &= \dim_{\mathbb{C}} D.
\end{aligned}
$$

This data determines a Shimura variety $M = \mathrm{Sh}(G,D)$, with a canonical model over $\mathbb{Q}$, where, for $K\subset G(\mathbb{A}_f)$ a compact open subgroup,

$$M_K(\mathbb{C}) \simeq G(\mathbb{Q})\backslash\Big(D\times G(\mathbb{A}_f)/K\Big). \tag{1.2}$$

Note that $D = D^+\cup D^-$ is a union of two copies of a bounded domain of type IV, [**69**], p.285. They are interchanged by the complex conjugation $w\mapsto\bar{w}$. If we let $G(\mathbb{R})^+$ be the subgroup of $G(\mathbb{R})$ which preserves D^+ and write

$$G(\mathbb{A}) = \coprod_j G(\mathbb{Q})G(\mathbb{R})^+ g_j K, \tag{1.3}$$

then

$$M_K(\mathbb{C}) \simeq \coprod_j \Gamma_j\backslash D^+, \tag{1.4}$$

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[2]Recall that, if $C(V) = C^+(V)\oplus C^-(V)$ is the Clifford algebra of V with its 2–grading, then there is a canonical embedding $V\hookrightarrow C^-(V)$, and

$$\mathrm{GSpin}(V) = \{\ g\in C^+(V)^\times \mid gg^\iota = \nu(g),\ \text{and } gVg^{-1}=V\ \},$$

where ι is the main involution of $C(V)$ and $\nu(g)$ is a scalar. There is an exact sequence

$$1\longrightarrow Z\longrightarrow G\longrightarrow SO(V)\longrightarrow 1$$

and the spinor norm homomorphism $\nu: G\to\mathbb{G}_m$. For $n\ge 1$, strong approximation holds for the semi-simple, simply connected group $\mathrm{Spin}(V)=\ker(\nu)$.

where $\Gamma_j = G(\mathbb{Q}) \cap G(\mathbb{R})^+ g_j K g_j^{-1}$. Thus, for general K, the quasi-projective variety M_K can have many components[3] and the individual components are only rational over some cyclotomic extension. The action of the Galois group on the components is described, for example, in [**17**], [**63**].

M_K is quasi-projective of dimension n over $\mathbb{Q}$, and projective if and only if the rational quadratic space V is anisotropic. By Meyer's Theorem, this can only happen for $n \leq 2$. In the range $3 \leq n \leq 5$, we can have $\text{witt}(V) = 1$, where witt(V) is the dimension of a maximal isotropic $\mathbb{Q}$-subspace of V. For $n \geq 6$, witt(V) $= 2$. A nice description of the Baily–Borel compactification of $\Gamma\backslash D^+$ and its toroidal desingularizations can be found in [**60**].

For small values of n, the M_K's include many classical varieties, for example:

$n = 1$, modular curves and Shimura curves, [**42**],
$n = 2$, Hilbert-Blumenthal surfaces and quaternionic versions, [**52**], [**76**],
$n = 3$, Siegel 3-folds and quaternionic analogues, [**54**], [**75**], [**25**],
$n \leq 19$, moduli spaces of K3 surfaces, [**5**].

Of course, such relations are discussed in many places, cf., for example, [**18**].

The most familiar example arises for the three dimensional rational quadratic space

$$V = \{\ x \in M_2(\mathbb{Q}) \mid \text{tr}(x) = 0\ \},$$

with $Q(x) = \det(x)$. In this case, $G = \text{GSpin}(V) = \text{GL}(2)$, the homomorphism $G \to SO(V) \simeq \text{PGL}(2)$ is defined via the conjugation action on V, $D \simeq \mathfrak{H}^+ \cup \mathfrak{H}^-$, the union of the upper and lower half planes, and the M_K's are the usual modular curves. Note that it is more convenient to work with GL(2) rather than PGL(2), and this leads to the choice of GSpin(V) rather than $SO(V)$ or the disconnected group $O(V)$ in general. More details about the case of signature $(1, 2)$ are given in the Appendix below.

More generally, one could consider quadratic spaces V over a totally real field $\boldsymbol{k}$ with $\text{sig}(V_{\infty_i}) = (n, 2)$ for $\infty_i \in S_1$ and $\text{sig}(V_{\infty_i}) = (n+2, 0)$ for $\infty_i \in S_2$ where $S_1 \cup S_2$ is a disjoint decomposition of the set of archimedean places of $\boldsymbol{k}$. If $S_2 \neq \emptyset$, then the varieties M_K are always projective. Such compact quotients are considered in [**47**], [**48**], and [**41**]. For a discussion of automorphic forms in this situation from a classical point of view, see [**71**].

[3] In fact, for $n \geq 1$, by strong approximation,

$$\pi_0(M_K) \simeq \mathbb{Q}^\times \mathbb{R}_+^\times \backslash \mathbb{A}^\times / \nu(K),$$

where $\nu(K)$ is the image of K under the norm map ν.

§2. Algebraic cycles.

An attractive feature of this family of Shimura varieties is that they have many algebraic cycles; in fact, there are sub-Shimura varieties of the same type of all codimensions. These can be constructed as follows.

Let $\mathcal{L}_D$ be the homogeneous line bundle over D with

$$\mathcal{L}_D \setminus \{0\} = \{w \in V(\mathbb{C}) \mid (w,w) = 0,\ (w,\bar{w}) < 0\}, \tag{2.1}$$

so that $\mathcal{L}_D$ is the restriction to D of the bundle $\mathcal{O}(-1)$ on $\mathbb{P}(V(\mathbb{C}))$. We equip $\mathcal{L}_D$ with the hermitian metric $||\ ||$ given by $||w||^2 = |(w,\bar{w})|$. The action of $G(\mathbb{R})$ on D lifts in a natural way to an action on $\mathcal{L}_D$, and hence, this bundle descends to a line bundle $\mathcal{L}$ on the Shimura variety M. For example, for a given compact open subgroup K, $\mathcal{L}_K \to M_K$, has a canonical model over $\mathbb{Q}$, [**32**], [**63**], and

$$\mathcal{L}_K(\mathbb{C}) \simeq G(\mathbb{Q})\backslash\Big(\mathcal{L}_D \times G(\mathbb{A}_f)/K\Big). \tag{2.2}$$

Any rational vector $x \in V(\mathbb{Q})$ defines a section s_x over D of the dual bundle $\mathcal{L}_D^\vee$ by the formula

$$(s_x, w) = (x, w), \tag{2.3}$$

and, for $x \neq 0$, the (possibly empty) divisor[4] in D of this section is given by

$$\operatorname{div}(s_x) = \{w \in D \mid (x,w) = 0\}/\mathbb{C}^\times =: D_x \subset D. \tag{2.4}$$

Assuming that $Q(x) := \frac{1}{2}(x,x) > 0$ and setting

$$V_x = x^\perp \tag{2.5}$$

and

$$G_x = \mathrm{GSpin}(V_x) = \text{stabilizer of } x \text{ in } G,$$

there is a sub-Shimura variety

$$Z(x) : \mathrm{Sh}(G_x, D_x) \longrightarrow \mathrm{Sh}(G,D) = M \tag{2.6}$$

giving a divisor $Z(x)_K$, rational over $\mathbb{Q}$, on M_K for each K.

[4] a rational quadratic divisor in Borcherds' terminology, [**3**].

If $Q(x) \leq 0$, and $x \neq 0$, then the section s_x is never zero on D, so that $D_x = \emptyset$. If $x = 0$, then we formally set $D_x = D$ and take $Z(0) = M$.

More generally, given an r–tuple of vectors $x \in V(\mathbb{Q})^r$ define V_x, G_x, D_x by the same formulas. If the span $\underline{x}$ of the components of x has dimension $r(x)$ and if the matrix

$$Q(x) = \frac{1}{2}\big((x_i, x_j)\big) \tag{2.7}$$

is positive semidefinite of rank $r(x)$, then the restriction of $(\ ,\ )$ to V_x has signature $(n - r(x), 2)$, and there is a corresponding cycle $Z(x) : Sh(G_x, D_x) \to Sh(G, D) = M$, of codimension $r(x) = \mathrm{rk}(Q(x)) \leq r$. If the rank of $Q(x)$ is less than $r(x)$ or if $Q(x)$ is not positive semi-definite, then $Z(x) = \emptyset$, [**47**], [**48**], [**49**].

For $g \in G(\mathbb{A}_f)$, we can also make a 'translated' cycle $Z(x, g)$ where, at level K,

(2.8)

$$\begin{aligned} Z(x, g; K) : G_x(\mathbb{Q})\backslash\Big(D_x \times G_x(\mathbb{A}_f)/K_x^g\Big) &\longrightarrow G(\mathbb{Q})\backslash\Big(D \times G(\mathbb{A}_f)/K\Big) = M_K(\mathbb{C}), \\ G_x(\mathbb{Q})(z, h)K_x^g \quad &\mapsto \quad G(\mathbb{Q})(z, hg)K. \end{aligned}$$

where we write $K_x^g = G_x(\mathbb{A}_f) \cap gKg^{-1}$ for short. This cycle is again rational over $\mathbb{Q}$.

Finally, we form certain weighted combinations of these cycles, essentially by summing over integral x's with a fixed matrix of inner products, [**41**]. More precisely, suppose that a K-invariant Schwartz function[5] $\varphi \in S(V(\mathbb{A}_f)^r)^K$ on r copies of the finite adeles $V(\mathbb{A}_f)$ of V and $T \in \mathrm{Sym}_r(\mathbb{Q})_{\geq 0}$ are given. Let

$$\Omega_T = \{\ x \in V^r \mid Q(x) = T\ \}, \tag{2.9}$$

and, assuming that $\Omega_T(\mathbb{Q})$ is nonempty, fix an element $x \in \Omega_T(\mathbb{Q})$ and write

$$\Omega_T(\mathbb{A}_f) \cap \mathrm{supp}(\varphi) = \coprod_j K g_j^{-1} x \tag{2.10}$$

for $g_j \in G(\mathbb{A}_f)$. Note that the sum is finite since the K-orbits give an open cover of the compact set $\Omega_T(\mathbb{A}_f) \cap \mathrm{supp}(\varphi)$. Then there is a cycle $Z(T, \varphi; K)$ in M_K defined by

$$Z(T, \varphi; K) = \sum_j \varphi(g_j^{-1} x)\, Z(x, g_j; K) \tag{2.11}$$

[5]For example, for $r = 1$, φ might be the characteristic function of the closure in $V(\mathbb{A}_f)$ of a coset $\mu + L$ of a lattice $L \subset V$.

of codimension $\text{rank}(T) =: r(T)$, given by a weighted combination of the $Z(x)$'s for x with $Q(x) = T$.

These weighted cycles have nice properties, [**41**]. For example, if $K' \subset K$ and $\text{pr} : M_{K'} \to M_K$ is the corresponding covering map, then

$$\text{pr}^* Z(T, \varphi; K) = Z(T, \varphi; K'). \tag{2.12}$$

Thus it is reasonable to drop K from the notation and write simply $Z(T, \varphi)$.

Example. The classical Heegner divisors, traced down to $\mathbb{Q}$, arise in the case $n = 1$, $r = 1$. A detailed description is given in Appendix I below.

§3. Modular generating functions.

In this section, we discuss the generating functions which can be constructed from the cycles $Z(T, \varphi)$, by taking their classes either in cohomology or in Chow groups. The main goal is to prove that such generating functions are, in fact, modular forms. Of course, these constructions are modeled on the work of Hirzeburch and Zagier [**36**] on generating functions for the cohomology classes of curves on Hilbert-Blumenthal surfaces.

3.1. Classes in cohomology.

The cycles defined above are very special cases of the locally symmetric cycles in Riemannian locally symmetric spaces studied some time ago in a long collaboration with John Millson, [**47**], [**48**], [**49**]. The results described in this section are from that joint work. For $T \in \text{Sym}_r(\mathbb{Q})_{\geq 0}$ and a weight function φ, there are cohomology classes

$$[\, Z(T, \varphi)\,] \in H^{2r(T)}(M_K) \qquad \text{and} \qquad [\, Z(T, \varphi)\,] \cup [\mathcal{L}^\vee]^{r - r(T)} \in H^{2r}(M_K), \tag{3.1}$$

where $r(T)$ is the rank of T and $[\mathcal{L}^\vee] \in H^2(M_K)$ is the cohomology class of the dual $\mathcal{L}^\vee$ of the line bundle $\mathcal{L}$. Here we view our cycles as defining linear functionals on the space of compactly supported closed forms, and hence these classes lie in the absolute cohomology $H^\bullet(M_K)$ of $M_K(\mathbb{C})$ with complex coefficients.

In [**49**], we proved:

Theorem 3.1. *For $\tau = u + iv \in \mathfrak{H}_r$, the Siegel space of genus r, the holomorphic function*

$$\phi_r(\tau, \varphi) = \sum_{T \in \text{Sym}_r(\mathbb{Q})_{\geq 0}} [\, Z(T, \varphi)\,] \cup [\mathcal{L}^\vee]^{r - r(T)}\, q^T,$$

is a Siegel modular form of genus r and weight $\frac{n}{2}+1$ valued in $H^{2r}(M_K)$.
Here $q^T = e(\mathrm{tr}(T\tau))$.

Idea of Proof. The main step is to construct a theta function *valued in the closed (r,r)–forms on $M_K(\mathbb{C})$*. Let $A^{(r,r)}(D)$ be the space of smooth (r,r)–forms on D, and let $S(V(\mathbb{R})^r)$ be the Schwartz space of $V(\mathbb{R})^r$. The group $G(\mathbb{R})$ acts naturally on both of these spaces. For $\tau \in \mathfrak{H}_r$, there is a Schwartz form, [**47**], [**48**], [**49**],

$$\varphi^r_\infty(\tau) \in \Big[S(V(\mathbb{R})^r) \otimes A^{(r,r)}(D) \Big]^{G(\mathbb{R})} \tag{3.2}$$

with the key property that, for all $x \in V(\mathbb{R})^r$, $d\varphi^r_\infty(\tau,x)=0$, i.e., $\varphi^r_\infty(\tau,x)$ is a closed form on D. Note that, for $g \in G(\mathbb{R})$ and $x \in V(\mathbb{R})^r$, the $G(\mathbb{R})$–invariance in (3.2) means that

$$g^*\varphi^r_\infty(\tau,x) = \varphi^r_\infty(\tau, g^{-1}x). \tag{3.3}$$

Thus, for example, for a fixed x, $\varphi_\infty(\tau,x) \in A^{(r,r)}(D)^{G(\mathbb{R})_x}$ is a closed $G(\mathbb{R})_x$–invariant form on D. Note that $\varphi^r_\infty(\tau)$ is *not* holomorphic in τ. For any $\varphi \in S(V(\mathbb{A}_f)^r)^K$, the Siegel theta function

$$\theta_r(\tau,\varphi)(\cdot,h) := \sum_{x \in V(\mathbb{Q})^r} \varphi^r_\infty(\tau,x)\varphi(h^{-1}x) \tag{3.4}$$

defines a closed (r,r)–form on $M_K(\mathbb{C})$. By the standard argument based on Poisson summation, which is explained in a little more detail in Appendix II, $\theta_r(\tau,\varphi)$ is modular of weight $\frac{n}{2}+1$ for a subgroup $\Gamma' \subset \mathrm{Sp}_r(\mathbb{Z})$, depending on φ. Finally, the cohomology class

$$\phi_r(\tau,\varphi) = [\theta_r(\tau,\varphi)] \tag{3.5}$$

of the theta form (3.4) coincides with the *holomorphic* generating function of the Theorem and hence this generating function is also modular of weight $\frac{n}{2}+1$. □

The Schwartz forms satisfy the cup product identity:

$$\varphi^{r_1}_\infty(\tau_1) \wedge \varphi^{r_2}_\infty(\tau_2) = \varphi^{r_1+r_2}_\infty(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}), \tag{3.6}$$

where the left side is an element of the space $S(V(\mathbb{R})^{r_1}) \otimes S(V(\mathbb{R})^{r_2}) \otimes A^{(r,r)}(D)$, with $r = r_1 + r_2$, and $\tau_j \in \mathfrak{H}_{r_j}$. Hence, for weight functions $\varphi_j \in S(V(\mathbb{A}_f)^{r_j})$, one has the identity for the theta forms

$$\theta_{r_1}(\tau_1,\varphi_1) \wedge \theta_{r_2}(\tau_2,\varphi_2) = \theta_r(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix}, \varphi_1 \otimes \varphi_2). \tag{3.7}$$

Passing to cohomology, (3.7) yields the pleasant identity, [**41**]:

$$\phi_{r_1}(\tau_1,\varphi_1)\cup\phi_{r_2}(\tau_2,\varphi_2)=\phi_{r_1+r_2}\big(\begin{pmatrix}\tau_1 & \\ & \tau_2\end{pmatrix},\varphi_1\otimes\varphi_2\big), \tag{3.8}$$

for the cup product of the generating functions valued in $H^\bullet(M_K)$. Comparing coefficients, we obtain the following formula for the cup product of our classes.
Suppose that $T_1\in \mathrm{Sym}_{r_1}(\mathbb{Q})_{>0}$ and $T_2\in \mathrm{Sym}_{r_2}(\mathbb{Q})_{>0}$. Then

$$[Z(T_1,\varphi_1)]\cup[Z(T_1,\varphi_2)]=\sum_{\substack{T\in \mathrm{Sym}_r(\mathbb{Q})_{\ge 0}\\ T=\begin{pmatrix}T_1 & * \\ * & T_2\end{pmatrix}}}[Z(T,\varphi_1\otimes\varphi_2)]\cup[\mathcal{L}^\vee]^{r-\mathrm{rk}(T)}. \tag{3.9}$$

3.2. Classes in Chow groups.
We can also take classes of the cycles in the usual Chow groups[6]. For this, when V is anisotropic so that M_K is compact, we consider the classes

$$\{\,Z(T,\varphi)\,\}\in \mathrm{CH}^{r(T)}(M_K)\qquad\text{and}\qquad \{\,Z(T,\varphi)\,\}\cdot\{\mathcal{L}^\vee\}^{r-r(T)}\in \mathrm{CH}^r(M_K) \tag{3.10}$$

in the Chow groups of M_K, and corresponding generating functions

$$\phi_r^{\mathrm{CH}}(\tau,\varphi)=\sum_{T\in \mathrm{Sym}_r(\mathbb{Q})_{\ge 0}}\{\,Z(T,\varphi)\,\}\cdot\{\mathcal{L}^\vee\}^{r-r(T)}\,q^T \tag{3.11}$$

valued in $\mathrm{CH}^r(M_K)_{\mathbb{C}}$. Here $\cdot$ denotes the product in the Chow ring $\mathrm{CH}^\bullet(M_K)$, and $\{\mathcal{L}^\vee\}\in \mathrm{Pic}(M_K)\simeq \mathrm{CH}^1(M_K)$ is the class of $\mathcal{L}^\vee$. Note that, for the cycle class map:

$$cl:\mathrm{CH}^r(M_K)\longrightarrow H^{2r}(M_K), \tag{3.12}$$

we have

$$\phi_r^{\mathrm{CH}}(\tau,\varphi)\mapsto \phi_r(\tau,\varphi), \tag{3.13}$$

so that the generating function $\phi_r^{\mathrm{CH}}(\tau,\varphi)$ 'lifts' the cohomology valued function $\phi_r(\tau,\varphi)$, which is modular by Theorem 3.1.

If V is isotropic, let $\widetilde{M}_K$ be a smooth toroidal compactification of M_K, [**60**]. Let $Y_K=\widetilde{M}_K\setminus M_K$ be the compactifying divisor, and let $\mathrm{CH}^1(\widetilde{M}_K,Y_K)$ be the quotient of $\mathrm{CH}^1(\widetilde{M}_K)$ by the subspace generated by the irreducible components of Y_K. We use the same notation for the classes $\{\,Z(T,\varphi)\,\}$ of our cycles in this group.
The following result is due to Borcherds, [**6**], [**7**].

[6] We only work with rational coefficients.

Theorem 3.2. *For $r=1$ and for a K-invariant weight function $\varphi \in S(V(\mathbb{A}_f))^K$, the generating function*

$$\phi_1^{\mathrm{CH}}(\tau,\varphi) = \{\mathcal{L}^\vee\}\,\varphi(0) + \sum_{t>0} \{\,Z(t,\varphi)\,\}\,q^t$$

is an elliptic modular form of weight $\frac{n}{2}+1$ valued in $\mathrm{CH}^1(\widetilde{M}_K, Y_K)$.

Proof. Since the result as stated is not quite in [**6**], we indicate the precise relation to Borcherds' formulation. For a lattice $L \subset V$ on which the quadratic form $Q(x) = \frac{1}{2}(x,x)$ is integer valued, let $L^\vee = \{x \in V \mid (x,L) \subset \mathbb{Z}\}$ be the dual lattice. Let $S_L \subset S(V(\mathbb{A}_f))$ be the finite dimensional subspace spanned by the characteristic functions φ_λ of the closures in $V(\mathbb{A}_f)$ of the cosets $\lambda + L$ where $\lambda \in L^\vee$. Every $\varphi \in S(V(\mathbb{A}_f))$ lies in some S_L for sufficiently small L. There is a (finite Weil) representation ρ_L of a central extension Γ' of $\mathrm{SL}_2(\mathbb{Z})$ on S_L. Suppose that F is a holomorphic function on $\mathfrak{H}$, valued in S_L, which is modular of weight $1-\frac{n}{2}$, i.e., for all $\gamma' \in \Gamma'$

$$F(\gamma'(\tau)) = j(\gamma',\tau)^{2-n}\rho_L(\gamma')F(\tau), \tag{3.14}$$

where $j(\gamma',\tau)$ with $j(\gamma',\tau)^2 = (c\tau+d)$ is the automorphy factor attached to γ', and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the projection of γ' to $\mathrm{SL}_2(\mathbb{Z})$. The function F is allowed to have a pole of finite order at ∞, i.e., F has a Fourier expansion of the form

$$F(\tau) = \sum_{\lambda \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\lambda(m)\,q^m\,\varphi_\lambda, \tag{3.15}$$

where only finitely many coefficients $c_\lambda(m)$ for $m<0$ can be nonzero. Note that, by the transformation law, $c_\lambda(m)$ can only be nonzero when $m \equiv -Q(\lambda) \mod \mathbb{Z}$. For any such F where, in addition, all $c_\lambda(-m)$ for $m \geq 0$ are in $\mathbb{Z}$, Borcherd constructs a meromorphic function $\Psi(F)$ on D with the properties, [**6**], [**7**], [**3**]:

(i) There is an integer N such that for any F, $\Psi(F)^N$ is a meromorphic automorphic form of weight $k = N\,c_0(0)/2$, i.e., a meromorphic section of $\mathcal{L}^{\otimes k}$, and

(ii)

$$\mathrm{div}(\Psi(F)^2) = \sum_{\lambda \in L^\vee/L} \sum_{m>0} c_\lambda(-m)\,Z(m,\varphi_\lambda). \tag{3.16}$$

In [**6**], Borcherds defines a rational vector space $\mathrm{CHeeg}(M_K)$ with generators $y_{m,\lambda}$, for $\lambda \in L^\vee/L$ and $m>0$ with $m \equiv Q(\lambda) \mod \mathbb{Z}$, and $y_{0,0}$ and relations

$$c_0(0)y_{0,0} + \sum_{\lambda} \sum_{m>0} c_\lambda(-m)\,y_{m,\lambda}, \tag{3.17}$$

as F runs over the quasi-modular forms of weight $1-\frac{n}{2}$, as above. Under the assumption that a certain space of vector valued forms has a basis with rational Fourier coefficients, Borcherds proved that the space $\mathrm{CHeeg}(M_K)$ is finite dimensional and that the generating function

$$\phi_1^B(\tau, L) = y_{0,0}\,\varphi_0^\vee + \sum_\lambda \sum_{m>0} y_{m,\lambda}\, q^m\, \varphi_\lambda^\vee, \tag{3.18}$$

valued in $\mathrm{CHeeg}(M_K)\otimes S_L^\vee$, is a modular form of weight $\frac{n}{2}+1$ for Γ'. Here $S_L^\vee$ is the dual space of S_L, with the dual representation $\rho_L^\vee$ of Γ'. William McGraw [**62**] recently proved that the necessary basis exists.

To finish the proof of our statement, we choose a nonzero (meromorphic) section Ψ_0 of $\mathcal{L}$ and define a map

$$\begin{aligned} \mathrm{CHeeg}(M_K) &\longrightarrow \mathrm{CH}(\widetilde{M}_K, Y_K), \\ y_{m,\lambda} &\mapsto Z(m,\varphi_\lambda) \\ y_{0,0} &\mapsto -\mathrm{div}(\Psi_0). \end{aligned} \tag{3.19}$$

This is well defined, since a relation is mapped to

$$-c_0(0)\,\mathrm{div}(\Psi_0) + \mathrm{div}(\Psi(F)^2) = N^{-1}\mathrm{div}(\Psi(F)^{2N}\,\Psi_0^{-2k}) \equiv 0, \tag{3.20}$$

since $\Psi(F)^{2N}\,\Psi_0^{-2k}$ is a meromorphic *function*[7] on M_K. Since the generating function $\phi_1(\tau,\varphi)$ is a finite linear combination of components of Borcherds' generating function, it is modular for some suitable subgroup of Γ', as claimed. Note that, $\{\mathcal{L}^\vee\} = \{-\mathrm{div}(\Psi_0)\}$. □

Problem 1: Is $\phi_r^{\mathrm{CH}}(\tau,\varphi)$ a Siegel modular form for $r>1$?

Problem 2: Does a cup product formula like (3.8) still hold?

Problem 3: Define classes $\{\,\widetilde{\mathcal{L}^\vee}\,\} \in \mathrm{Pic}(\widetilde{M}_K)$,

$$\{\,\widetilde{Z}(T,\varphi)\,\} \in \mathrm{CH}^{r(T)}(\widetilde{M}_K) \qquad \text{and} \qquad \{\,\widetilde{Z}(T,\varphi)\,\}\cdot\{\widetilde{\mathcal{L}^\vee}\}^{r-r(T)} \in \mathrm{CH}^r(\widetilde{M}_K)$$

so that the resulting generating function $\phi_r^{\mathrm{CH}}(\tau,\varphi)$ is modular.

Additional information about the map

$$\mathrm{CHeeg}(M_K)/\mathbb{Q}\,y_{0,0} \longrightarrow \mathrm{CH}(\widetilde{M}_K, Y_K)/\mathbb{Q}\{\widetilde{\mathcal{L}^\vee}\}, \tag{3.21}$$

e.g., concerning injectivity, was obtained by Bruinier, [**12**], [**13**].

[7]Of course, one needs to check that it extends to a meromorphic function on $\widetilde{M}_K$.

§4. Connections with values of Eisenstein series.

To obtain classical scalar valued modular forms, one can apply linear functionals to the modular generating functions valued in cohomology. For a moment, we again assume that we are in the case of compact quotient. Then, using the class

$$[\mathcal{L}^\vee] \in H^2(M_K), \tag{4.1}$$

and the composition

$$H^{2r}(M_K) \times H^{2(n-r)}(M_K) \longrightarrow H^{2n}(M_K) \overset{\deg}{\longrightarrow} \mathbb{C}, \tag{4.2}$$

of the cup product and the degree map, we have:

$$\deg(\,\phi_r(\tau,\varphi) \cup [\mathcal{L}^\vee]^{n-r}\,) = \int_{M_K} \theta_r(\tau,\varphi) \wedge \Omega^{n-r} := I_r(\tau,\varphi), \tag{4.3}$$

where Ω is the Chern form of the line bundle $\mathcal{L}^\vee$ for its natural metric.

Now, the Siegel–Weil formula, [**78**], [**51**], relates the integral $I_r(\tau,\varphi)$ of a theta function determined by a Schwartz function $\varphi \in S(V(\mathbb{A}_f)^r)^K$ to a special value of a Siegel Eisenstein series $E_r(\tau,s,\varphi)$, also associated to φ. The parameter s in this Eisenstein series is normalized as in Langlands, so that there is a functional equation with respect to $s \mapsto -s$, and the halfplane of absolute convergence is $\mathrm{Re}(s) > \frac{r+1}{2}$. Note that, to apply the Siegel–Weil formula, we must first relate the integral of the theta *form* occurring in (4.3) to the adèlic integral of the theta *function* occurring in the Siegel–Weil theory, [**44**], section 4. Hence, we obtain:

The volume formula. *In the case of compact quotient,* [**41**],

$$\deg(\,\phi_r(\tau) \cup [\mathcal{L}^\vee]^{n-r}\,) \overset{(1)}{=} I_r(\tau,\varphi) \overset{(2)}{=} \mathrm{vol}(M_K,\Omega^n) \cdot E_r(\tau,s_0,\varphi), \tag{4.4}$$

where

$$s_0 = \frac{n+1-r}{2}. \tag{4.5}$$

In fact, this formula should hold in much greater generality, i.e., when V is isotropic. First of all, the theta integral is termwise convergent whenever Weil's condition $r < n+1-\mathrm{witt}(V)$ holds, and so the identity (2) in (4.4) is then valid. The result of [**50**] can be applied and the argument given in [**44**] for the case $r = 1$ carries over to prove the following.

Theorem 4.1. *When $r < n+1-\mathrm{witt}(V)$, there is an identity*

$$\sum_{T\geq 0} \mathrm{vol}(Z(T,\varphi),\Omega^{n-r(T)})\, q^T = \mathrm{vol}(M,\Omega^n)\cdot E_r(\tau,s_0,\varphi).$$

It remains to give a cohomological interpretation of the left side of this identity in the noncompact case.

Some sort of regularization of the theta integral, say by the method of [**51**], is needed to obtain an extension of (2) to the range $r \geq n+1-\mathrm{witt}(V)$, i.e., to the cases $r = n-1$ and n when $\mathrm{witt}(V) = 2$ or the case $r = n$, if $\mathrm{witt}(V) = 1$. For example, in the case of modular curves, where $n = r = 1$, it was shown by Funke [**20**] that the theta integral coincides with Zagier's nonholomorphic Eisenstein series of weight $\frac{3}{2}$, [**82**]. In this case, there are definitely (non-holomorphic!) correction terms which do not have an evident cohomological meaning, although they are consistent with a suitable arithmetic Chow group formulation, cf. Yang's article, [**81**]. Recent work of Funke and Millson [**21**] considered the pairing of the theta form with closed forms not of compact support in the case of arithmetic quotients of hyperbolic n-space.

Examples:

1. If $n = 1$ and V is anistropic, so that $M = M_K$ is a Shimura curve over $\mathbb{Q}$, then

$$\mathrm{vol}(M_K)\cdot E_1(\tau,\frac{1}{2},\varphi) = \deg(\phi_1(\tau,\varphi)) = \mathrm{vol}(M_K,\Omega) + \sum_{t>0}\deg(Z(t,\varphi))\, q^t \tag{4.6}$$

is a special value at $s = \frac{1}{2}$ of an Eisenstein series of weight $\frac{3}{2}$, and the $Z(t,\varphi)$'s are Heegner type 0–cycles on M_K, cf. Appendix I. This identity is described in more detail in [**56**].

2. If $n = 2$ and V has $\mathrm{witt}(V) = 1$ or is anisotropic, so that M_K is a Hilbert–Blumenthal surface for some real quadratic field or a compact analogue, then

$$\mathrm{vol}(M_K)\cdot E_1(\tau,1,\varphi) = \deg(\phi_1(\tau,\varphi)\cup\Omega) = \mathrm{vol}(M_K) + \sum_{t>0}\mathrm{vol}(Z(t,\varphi),\Omega)\, q^t \tag{4.7}$$

is the special value at $s = 1$ of an Eisenstein series of weight 2, and the $Z(t,\varphi)$'s are Hirzebruch-Zagier type curves [**75**] on M_K.

3. If $n = 2$ and V is anisotropic, then

$$\begin{aligned}\mathrm{vol}(M_K)\cdot E_2(\tau,\frac{1}{2},\varphi) &= \deg(\phi_2(\tau,\varphi)) \\ &= \mathrm{vol}(M_K) + \sum_{\substack{T\in\mathrm{Sym}_2(\mathbb{Q})_{\geq 0}\\ r(T)=1}}\mathrm{vol}(Z(T,\varphi),\Omega)\, q^T + \sum_{T>0}\deg(Z(T,\varphi))\, q^T\end{aligned} \tag{4.8}$$

is the special value at $s = \frac{1}{2}$ of an Eisenstein series of weight 2 and genus 2, and, for $T > 0$, the $Z(T, \varphi)$'s are 0–cycles. Gross and Keating, [**28**], observed such a phenomenon in the split case as well.

4. If $n = 3$, and for V with $\text{witt}(V) = 2$, M_K is a Siegel modular 3-fold. Then, for $r = 1$, the $Z(t, \varphi)$'s are combinations of Humbert surfaces, and the identity of Theorem 4.1 asserts that their volumes are the Fourier coefficients of an Eisenstein series of weight $\frac{5}{2}$, [**76**], [**44**].

II

Speculations on the arithmetic theory

The main idea is that many of the phenomena described above have an analogue in arithmetic geometry, where the varieties M are replaced by integral models $\mathcal{M}$ over $\text{Spec}\,(\mathbb{Z})$, the cycles $Z(T, \varphi)$ are replaced by arithmetic cycles on $\mathcal{M}$, and the classes of these cycles are taken in arithmetic Chow groups $\widehat{\text{CH}}^r(\mathcal{M})$, [**24**], [**74**]. One could then define a function $\widehat{\phi}_r$ valued in $\widehat{\text{CH}}^r(\mathcal{M})$, lifting the modular generating function ϕ_r valued in cohomology. The main goal would be to prove the modularity of $\widehat{\phi}_r$ and to find analogues of the identities discussed above, where the values of the Eisenstein series occurring in section 4 are replaced by their derivatives, i.e., by the second terms in their Laurent expansions.

At this point, I am going to give an idealized picture which ignores many serious technical problems involving: (i) the existence of good integral models, (ii) bad reduction and the possible bad behavior of cycles at such places, (iii) noncompactness, boundary contributions, (iv) extensions of the Gillet-Soulé theory [**24**] of arithmetic Chow groups $\widehat{\text{CH}}^r(\mathcal{M})$ to allow singular metrics, (cf, Bost, [**9**], Kühn, [**58**], Burgos-Kramer-Kühn, [**15**]), (iv) suitable definitions of Green functions, etc., etc.[8]

Nevertheless, the idealized picture can serve as a guide and, with sufficient effort, one can obtain rigorous results in various particular cases, [**42**], [**52**], [**53**], [**54**], [**55**], [**56**]. In all of these cases, we only consider a good maximal compact subgroup K and a specific weight function φ determined by a nice lattice, so, in the discussion to follow, we will suppress both K and φ from the notation.

[8]The fastidious reader may want to stop here.

§5. Integral models and cycles.

Suppose that we have:

(5.1)
$$\begin{aligned} \mathcal{M} &= \text{ a regular model of } M \text{ over Spec}\,(\mathbb{Z}),\\ \widehat{\mathrm{CH}}^{\bullet}(\mathcal{M}) &= \text{its (extended) arithmetic Chow groups},\\ \widehat{\omega} &= \text{ extension of the metrized line bundle } \mathcal{L}^{\vee} \text{ to } \mathcal{M},\\ \widehat{\omega} &\in \widehat{\mathrm{Pic}}(\mathcal{M}) \simeq \widehat{\mathrm{CH}}^{1}(\mathcal{M})\\ \mathcal{Z}(T) &= \text{ an extension of } Z(T) \text{ on } M \text{ to a cycle on } \mathcal{M}, \text{ so that} \end{aligned}$$

$$\begin{array}{cccccc} & \mathcal{Z}(T) & \longrightarrow & \mathcal{M} & & \\ & \uparrow & & \uparrow & & \\ Z(T) = & \mathcal{Z}(T)_{\mathbb{Q}} & \longrightarrow & \mathcal{M}_{\mathbb{Q}} & = M & \qquad \text{(generic fibers).} \end{array}$$

Finally, to obtain classes in the Gillet–Soulé arithmetic Chow groups $\widehat{\mathrm{CH}}^{r}(\mathcal{M})$ from the $\mathcal{Z}(T)$'s we need Green forms, [**24**], [**11**], [**8**]. Based on the constructions for $r = 1$, [**42**], and for $r = 2$, $n = 1$, [**43**], we suppose that these have the following form:

(5.2)
$$\begin{aligned} \tau &= u + iv \in \mathfrak{H}_r\\ \Xi(T, v) &= \text{ Green form for } Z(T), \text{ depending on } v,\\ \widehat{\mathcal{Z}}(T, v) &= (\mathcal{Z}(T), \Xi(T, v)) \in \widehat{\mathrm{CH}}^{r(T)}(\mathcal{M}). \end{aligned}$$

In all cases done so far, $0 \le n \le 3$, [**55**], [**42**], [**53**], [**52**], [**54**], M is of PEL type and the model $\mathcal{M}$ is obtained by extending the moduli problem over $\mathbb{Q}$ to a moduli problem over Spec$\,(\mathbb{Z})$ or, at least, Spec$\,(\mathbb{Z}[N^{-1}])$ for a suitable N. The cycles $\mathcal{Z}(T)$ are defined by imposing additional endomorphisms satisfying various compatibilities, the special endomorphisms, cf. [**43**], [**45**] for further discussion.

With such a definition, it can happen that $\mathcal{Z}(T)$ is non-empty, even when the generic fiber $\mathcal{Z}(T)_{\mathbb{Q}} = Z(T)$ is empty. For example, purely vertical divisors can occur in the fibers of bad reduction of the arithmetic surfaces attached to Shimura curves, [**53**]. In addition, there can be cases where $\mathcal{Z}(T)$ is empty, but $\Xi(T, v)$ is a nonzero smooth form on $M(\mathbb{C})$, so that there are classes

(5.3)
$$\widehat{\mathcal{Z}}(T, v) = (0, \Xi(T, v)) \in \widehat{\mathrm{CH}}^{r}(\mathcal{M})$$

'purely vertical at infinity, even for T not positive semi-definite, [**42**], [**43**].

Finally, we define the **arithmetic theta function:**

$$\widehat{\phi}_r(\tau) = \sum_{T \in \mathrm{Sym}_r(\mathbb{Q})} \widehat{\mathcal{Z}}(T, v) \cdot \widehat{\omega}^{r-r(T)}\, q^T \in \widehat{\mathrm{CH}}^r(\mathcal{M}), \tag{5.4}$$

where $\cdot$ denotes the product in the arithmetic Chow ring $\widehat{\mathrm{CH}}^\bullet(\mathcal{M})$. Note that this function is not holomorphic in τ, since the Green forms depend on v. Under the restriction maps

$$\mathrm{res} : \widehat{\mathrm{CH}}^r(\mathcal{M}) \longrightarrow \mathrm{CH}^r(\mathcal{M}_\mathbb{Q}) \longrightarrow H^{2r}(M), \tag{5.5}$$

we have

$$\widehat{\phi}_r(\tau) \mapsto \phi_r^{\mathrm{CH}}(\tau) \mapsto \phi_r(\tau), \tag{5.6}$$

so that $\widehat{\phi}_r$ lifts ϕ_r^{CH} and ϕ_r to the arithmetic Chow group.

Problem 4: Can the definitions be made so that $\widehat{\phi}_r(\tau)$ is a Siegel modular form of weight $\frac{n}{2} + 1$ valued in $\widehat{\mathrm{CH}}^r(\mathcal{M})$, lifting ϕ_r and ϕ_r^{CH}?

At present, this seems out of reach, especially for $1 < r < n+1$.

Problem 5: Is there an intersection product formula for the arithmetic Chow ring:

$$\widehat{\phi}_{r_1}(\tau_1) \cdot \widehat{\phi}_{r_2}(\tau_2) \overset{??}{=} \widehat{\phi}_{r_1+r_2}\left(\begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix} \right) \tag{5.7}$$

lifting the cup product relation (3.8) in cohomology?

§6. Connections with derivatives of Eisenstein series.

As in the standard Gillet–Soulé theory, suppose[9] that there is an arithmetic degree map

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{n+1}(\mathcal{M}) \longrightarrow \mathbb{C}, \tag{6.1}$$

and a height pairing

$$\langle\ ,\ \rangle : \widehat{\mathrm{CH}}^r(\mathcal{M}) \times \widehat{\mathrm{CH}}^{n+1-r}(\mathcal{M}) \longrightarrow \mathbb{C}, \qquad \langle \widehat{\mathcal{Z}}_1, \widehat{\mathcal{Z}}_2 \rangle = \widehat{\deg}(\widehat{\mathcal{Z}}_1 \cdot \widehat{\mathcal{Z}}_2). \tag{6.2}$$

These can be used to produce 'numerical' generating functions from the $\widehat{\phi}_r$'s.

[9] Recall that, in the noncompact cases, we have to use some extended theory of arithmetic Chow groups, [**15**], which allows the singularities of the natural metric on $\widehat{\omega}$.

Let

$$\mathcal{E}_r(\tau, s) = C(s)\, E_r(\tau, s, \varphi_0) \tag{6.3}$$

be the Siegel–Eisenstein series of weight $\frac{n}{2}+1$ and genus r associated to φ_0, our standard weight function, with suitably normalizing factor $C(s)$, cf. [**56**] for an example of this normalization. The choice of $C(s)$ becomes important in the cases in which the leading term in nonzero. Then the following **arithmetic volume formula** is an analogue of the volume formula of Theorem 4.1 above:

Problem 6: For a suitable definition of $\mathcal{E}_r(\tau, s)$, show that

$$\begin{aligned} \mathcal{E}'_r(\tau, s_0) &\overset{??}{=} \langle\, \widehat{\phi}_r(\tau), \widehat{\omega}^{n+1-r} \,\rangle \\ &= \sum_T \widehat{\deg}\,\big(\, \widehat{\mathcal{Z}}(T, v)\cdot \widehat{\omega}^{n+1-r(T)} \,\big)\, q^T. \end{aligned} \tag{6.4}$$

where $s_0 = \frac{n+1-r}{2}$ is the critical value of s occurring in the Siegel–Weil formula. Here r lies in the range $1 \le r \le n+1$.

Remarks:
(i) The identity (6.4) can be proved without knowing that $\widehat{\phi}_r$ is modular, and one can obtain partial results by identifying corresponding Fourier coefficients on the two sides.
(ii) One can view the quantities $\widehat{\deg}\,(\,\widehat{\mathcal{Z}}(T, v)\cdot\widehat{\omega}^{n+1-r(T)}\,)$ as arithmetic volumes or heights, [**11**].
(iii) Assuming that $C(s_0) = \mathrm{vol}(M)$, the leading term

$$\mathcal{E}_r(\tau, s_0) = \mathrm{vol}(M)\, E_r(\tau, s_0) \tag{6.5}$$

of the normalized Eisenstein series at $s = s_0$ is just the generating function for geometric volumes, via Theorem 4.1.
(iv) In the case $r = n+1$, so that $\widehat{\phi}_r(\tau) \in \widehat{\mathrm{CH}}^{n+1}(\mathcal{M})$, the image of $\widehat{\phi}_{n+1}$ in cohomology or in the usual Chow ring of $\mathcal{M}_{\mathbb{Q}}$ is identically zero, since this group vanishes. On the other hand, the Eisenstein series $E_{n+1}(\tau, s)$ is incoherent in the sense of [**42**], [**43**], the Siegel–Weil point is $s_0 = 0$, and $E_{n+1}(\tau, 0)$ is also identically zero. Thus the geometric volume identity is trivially valid. The arithmetic volume formula would then be

$$\widehat{\deg}\,(\widehat{\phi}_{n+1}(\tau)) \overset{??}{=} \mathcal{E}'_{n+1}(\tau, 0). \tag{6.6}$$

Examples:

1. Moduli of CM elliptic curves. $n = 0$, $r = 1$, [**55**]. Here V is a negative definite binary quadratic form given by the negative of the norm form of an imaginary quadratic field $\boldsymbol{k}$, and $\mathcal{M}$ is the moduli stack of elliptic curves with CM by $O_{\boldsymbol{k}}$, the ring of integers of $\boldsymbol{k}$. For $t \in \mathbb{Z}_{>0}$, the cycle $\mathcal{Z}(t)$ is either empty or is a 0-cycle supported in a fiber $\mathcal{M}_p$ for a prime p determined by t. The identity

$$\widehat{\deg}\,(\widehat{\phi}_1(\tau)) = \mathcal{E}_1'(\tau, 0) \tag{6.7}$$

for the central derivative of an incoherent Eisenstein series of weight 1 is proved in [**55**], in the case in which $\boldsymbol{k}$ has prime discriminant. The computation of the arithmetic degrees is based on the result of Gross, [**26**], which is also the key to the geometric calculations in [**31**].

Remark: In the initial work on the arithmetic situation [**42**] and in the subsequent joint papers with Rapoport, [**54**] and [**52**], the main idea was to view the central derivative of the incoherent Eisenstein series, restricted to the diagonal, as giving the height pairing of cycles in complementary degrees, cf. formula (6.10) below. At the Durham conference in 1996, Gross insisted that it would be interesting to consider the 'simplest case', i.e., $n = 0$. Following his suggestion, we obtained the results of [**55**] and came to see that the central derivative should *itself* have a nice geometric interpretation, as a generating function for the arithmetic degrees of 0–cycles on $\mathcal{M}$, without restriction to the diagonal. This was a crucial step in the development of the picture discussed here.

2. Curves on arithmetic surfaces. $n = 1$, $r = 1$. Here, if V is the space of trace zero elements of an indefinite division quaternion algebra over $\mathbb{Q}$, $\mathcal{M}$ is the arithmetic surface associated to a Shimura curve. For $t \in \mathbb{Z}_{>0}$, the cycle $\mathcal{Z}(t)$ is a divisor on $\mathcal{M}$ and can have vertical components. The identity

$$\langle\, \widehat{\phi}_1(\tau), \widehat{\omega}\,\rangle = \mathcal{E}_1'(\tau, \frac{1}{2}) \tag{6.8}$$

is proved in [**56**]. Here $\mathcal{E}_1(\tau, s)$ is a normalized Eisenstein series of weight $\frac{3}{2}$. An unknown constant, conjectured to be zero, occurs in the definition of the class $\widehat{\mathcal{Z}}(0, v)$ in the constant term of the generating function. This constant arises because we do not have, at present, an explicit formula for the quantity $\langle\widehat{\omega}, \widehat{\omega}\rangle$ for the arithmetic surface attached to a Shimura curve. In the analogous example for modular curves, discussed in Yang's talk, [**81**], the quantity $\langle\widehat{\omega}, \widehat{\omega}\rangle$ is known, thanks to the work of Ulf Kühn [**58**] and Jean-Benoit Bost [**9**], [**10**], independently. The computation of such arithmetic invariants via an arithmetic Lefschetz formula is discussed in [**61**]. The identity (6.8) is the first arithmetic case in which the critical point s_0 for $\mathcal{E}(\tau, s)$ is not zero and and the leading term $\mathcal{E}(\tau, \frac{1}{2})$ does not vanish. It is also the first case in which a truely global quantity, the pairing $\langle\, \widehat{\mathcal{Z}}(t, v), \widehat{\omega}\,\rangle$ for a horizontal cycle $\mathcal{Z}(t)$, must be computed; it is determined as the Faltings height of a CM elliptic curve, [**56**].

3. 0–cycles on arithmetic surfaces. In the case $n = 1$, $r = 2$, $\widehat{\phi}_2(\tau)$ is a generating function for 0–cycles on the arithmetic surface $\mathcal{M}$. The combination of [**42**], joint work with

Rapoport [**53**], and current joint work with Rapoport and Yang, [**57**], comes very close to proving the identity

$$\widehat{\deg}\,(\widehat{\phi}_2(\tau)) \overset{??}{=} \mathcal{E}_2'(\tau,0), \tag{6.9}$$

again up to an ambiguity in the constant term of the generating function due to the lack of a formula for $\langle\widehat{\omega},\widehat{\omega}\rangle$. For $T>0$ and p-regular, as defined in [**43**], the cycle $\mathcal{Z}(T)$ is a 0-cycle concentrated in a single fiber $\mathcal{M}_p$ for a prime p determined by T. In this case, the computation of $\widehat{\deg}(\mathcal{Z}(T))$ amounts to a counting problem and a problem in the deformation theory of p-divisible groups. The latter is a special case of a deformation problem solved by Gross and Keating [**28**]. On the analytic side, the computation of the corresponding term in the central derivative of the Eisenstein series amounts to the *same* counting problem and the computation of the central derivative of a certain Whittaker function on $\mathrm{Sp}_2(\mathbb{Q}_p)$. This later computation depends on the explicit formulas due to Kitaoka [**38**] for the representation densities of T by unimodular quadratic forms of rank $4+2j$, cf. [**43**], section 5, for a more detailed discussion.

4. Siegel modular varieties. $n=3$, [**54**]. The Shimura variety M attached to a rational quadratic space V of signature $(3,2)$ is, in general, a 'twisted' version of a Siegel 3-fold. The canonical model M over $\mathbb{Q}$ can be obtained as a moduli space of polarized abelian varieties of dimension 16 with an action of a maximal order O_C in the Clifford algebra of V. A model $\mathcal{M}$ over $\mathrm{Spec}\,(\mathbb{Z}[N^{-1}])$, for a suitable N can likewise be defined as a moduli space, [**54**]. The possible generating functions and their connections with Eisenstein series are given in the following chart, [**45**]:

$r=1,$	$\mathcal{Z}(t)_{\mathbb{Q}} =$ Humbert surface,	$\hat{\phi}_1(\tau) = \hat{\omega} + ? + \sum_{t\neq 0} \widehat{\mathcal{Z}}(t,v)\,q^t,$	$\langle\hat{\phi}_1(\tau),\hat{\omega}^3\rangle \overset{?}{=} \mathcal{E}_1'(\tau,\frac{3}{2})$
$r=2,$	$\mathcal{Z}(t)_{\mathbb{Q}} =$ curve	$\hat{\phi}_2(\tau) = \hat{\omega}^2 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\,q^T,$	$\langle\hat{\phi}_2(\tau),\hat{\omega}^2\rangle \overset{?}{=} \mathcal{E}_2'(\tau,1)$
$r=3,$	$\mathcal{Z}(T)_{\mathbb{Q}} =$ 0–cycle,	$\hat{\phi}_3(\tau) = \hat{\omega}^3 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\,q^T,$	$\langle\hat{\phi}_2(\tau),\hat{\omega}\rangle \overset{?}{=} \mathcal{E}_3'(\tau,\frac{1}{2})$
$r=4,$	$\mathcal{Z}(T)_{\mathbb{Q}} = \emptyset,$	$\hat{\phi}_4(\tau) = \hat{\omega}^4 + ? + \sum_{T\neq 0} \widehat{\mathcal{Z}}(T,v)\,q^T,$	$\widehat{\deg}\,\hat{\phi}_4(\tau) \overset{?}{=} \mathcal{E}_4'(\tau,0).$

The Siegel Eisenstein series $\mathcal{E}_r(\tau,s)$ and, conjecturally, the generating functions $\hat{\phi}_r(\tau)$ have weight $\frac{5}{2}$ and genus r, and the last column in the chart gives the 'arithmetic volume formula' of Problem 6 in each case. Some evidence for the last of these identities was obtained in joint work with M. Rapoport, [**54**]. In the case of a prime p of good reduction a model of M over $\mathrm{Spec}\,(\mathbb{Z}_p)$ is defined in [**54**], and cycles are defined by imposing special endomorphisms. For $r=4$, the main results of [**54**] give a criterion for $\mathcal{Z}(T)$ to be a 0–cycle in a fiber $\mathcal{M}_p$ and show that, when this is the case, then $\widehat{\deg}\,((\mathcal{Z}(T),0))\,q^T = \mathcal{E}_{4,T}'(\tau,0)$. The calculation of the left hand side is again based on the result of Gross and Keating, [**28**]. For $r=1$, the results of [**44**], cf., in particular, sections 5 and 6, are consistent with the identity in the first row, which involve arithmetic volumes of divisors.

5. Divisors. For any n, when $r=1$, the arithmetic volume formula predicts that the second term $\mathcal{E}'_1(\tau,\frac{n}{2})$ in the Laurent expansion of an elliptic modular Eisenstein series of weight $\frac{n}{2}+1$ at the point $s_0=\frac{n}{2}$ has Fourier coefficients involving the arithmetic volumes $\langle \widehat{\mathcal{Z}}(t,v),\widehat{\omega}^n\rangle$ of divisors on the integral model $\mathcal{M}$ of M. The first term $\mathcal{E}_1(\tau,\frac{n}{2})$ in the Laurent expansion at this point has Fourier coefficients involving the usual volumes of the corresponding geometric cycles. For example, for a suitable choice of V, $\mathcal{E}(\tau,\frac{n}{2})$ is a familiar classical Eisenstein series, e.g., $E_2(\tau)$ (non-holomorphic), $E_4(\tau)$, $E_6(\tau)$, etc., for $\dim(V)$ even, and Cohen's Eisenstein series $E_{\frac{n}{2}+1}(\tau)$, [**16**], for $\dim(V)$ odd[10]. This means that the second term in the Laurent expansion of such classical Eisenstein series should contain information from arithmetic geometry! Again, related results are obtained in [**44**].

An Important Construction. We conclude this section with an important identity which relates the generating function for height pairings with that for arithmetic degrees. Suppose that $n=2r-1$ is odd. Then the various conjectural identities above, in particular (5.7) and (6.6), lead to the formula:

$$\begin{aligned}\langle \widehat{\phi}_r(\tau_1),\widehat{\phi}_r(\tau_2)\,\rangle &= \widehat{\deg}\,\big(\,\widehat{\phi_r}(\tau_1)\cdot\overline{\widehat{\phi}_r(\tau_2)}\,\big) \\ &\overset{??}{=} \widehat{\deg}\,(\widehat{\phi}_{2r}(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix})\,)\,) && \text{(via (5.7))} \\ &\overset{??}{=} \mathcal{E}'_{2r}(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix},0\,) && \text{(via (6.6))}\end{aligned} \tag{6.10}$$

relating the height pairing of the series $\widehat{\phi}_r(\tau)\in\widehat{\mathrm{CH}}^r(\mathcal{M})$ in the middle degree with the restriction of the central derivative of the Siegel Eisenstein series $\mathcal{E}_{2r}(\tau,s)$ of genus $2r$ and weight $r+\frac{1}{2}$. Note that this weight is always half-integral. These series are the 'incoherent' Eisenstein series discussed in [**42**] and [**43**]. The conjectural identity (6.10) will be used in an essential way in the next section.

III

Derivatives of L-series

In this section, we explain how the modularity of the arithmetic theta functions and the conjectural relations between their inner products and derivatives of Siegel Eisenstein series

[10] Here the subscript denotes the weight rather than the genus

might be connected with higher dimensional Gross–Zagier type formulas expressing central derivatives of certain L-functions in terms of height pairings of special cycles. These formulas should be analogues of those connecting *values* of certain L-functions to inner products of theta lifts, e.g., the Rallis inner product formula, [**68**], [**59**], [**51**], section 8, and Appendix II below.

§7. Arithmetic theta lifts.

Suppose that $f \in S^{(r)}_{\frac{n}{2}+1}$ is a holomorphic Siegel cusp form of weight $\frac{n}{2}+1$ and genus r for some subgroup $\Gamma' \subset \mathrm{Sp}_r(\mathbb{Z})$. Then, assuming the existence of the generating function $\widehat{\phi}_r$ valued in $\widehat{\mathrm{CH}}^r(\mathcal{M})$ and that this function is also modular for Γ', we can define an **arithmetic theta lift**:

$$\begin{aligned}(7.1)\qquad \hat{\theta}_r(f) :&= \langle\, f, \widehat{\phi}_r \,\rangle_{\mathrm{Pet}} \\ &= \int_{\Gamma'\backslash\mathfrak{H}_r} f(\tau)\overline{\widehat{\phi}_r(\tau)}\,\det(v)^{\frac{n}{2}+1}\,d\mu(\tau) \in \widehat{\mathrm{CH}}^r(\mathcal{M})\end{aligned}$$

where $\langle\ ,\ \rangle_{\mathrm{Pet}}$ is the Petersson inner product. Thus, we get a map

$$(7.2)\qquad S^{(r)}_{\frac{n}{2}+1} \longrightarrow \widehat{\mathrm{CH}}^r(\mathcal{M}), \qquad f \mapsto \hat{\theta}_r(f).$$

This map is an arithmetic analogue of a theta correspondence like the Shimura lift, [**70**], from forms of weight $\frac{3}{2}$ to forms of weight 2, which can be defined by integration against a classical theta function, [**64**]. For example, if f is a Hecke eigenform, then $\hat{\theta}_r(f)$ will also be an Hecke eigenclass.

§8. Connections with derivatives of L-functions.

Restricting to the case $n = 2r-1$, where the target is the arithmetic Chow group $\widehat{\mathrm{CH}}^r(\mathcal{M})$ in the middle dimension, we can compute the height pairing of the classes $\hat{\theta}_r(f)$ using identity (6.10) above:

$$(8.1)\qquad \begin{aligned}\langle\, \hat{\theta}_r(f_1), \hat{\theta}_r(f_2)\,\rangle &= \big\langle\, \langle f_1, \widehat{\phi}_r\rangle_{\mathrm{Pet}}, \langle f_2, \widehat{\phi}_r\rangle_{\mathrm{Pet}}\,\big\rangle \\ &= \big\langle\, f_1 \otimes \bar{f}_2, \langle \widehat{\phi}_r(\tau_1), \widehat{\phi}_r(\tau_2)\rangle\,\big\rangle_{\mathrm{Pet}} \\ &\overset{??}{=} \big\langle\, f_1 \otimes \bar{f}_2, \mathcal{E}'_{2r}\big(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix}, 0\big)\,\big\rangle_{\mathrm{Pet}} \qquad \text{by (6.10)} \\ &= \frac{\partial}{\partial s}\Big\{ \big\langle\, f_1 \otimes \bar{f}_2, \mathcal{E}_{2r}\big(\begin{pmatrix}\tau_1 & \\ & -\bar{\tau}_2\end{pmatrix}, \bar{s}\big)\,\big\rangle_{\mathrm{Pet}}\Big\}\Big|_{s=0}\end{aligned}$$

Here we use the hermitian extension of the height pairing (6.2) to $\widehat{\mathrm{CH}}^1(\mathcal{M})_{\mathbb{C}}$ taken to be conjugate linear in the second argument. Aficionados of Rankin–Selberg integrals will now recognize the **doubling integral** of Rallis and Piatetski-Shapiro, [**65**], and, in classical language, of Böcherer, [**1**], and Garrett, [**22**], in the last line of (8.1). If f_1 and f_2 correspond to different irreducible cuspidal automorphic representations, then the integral in (8.1) vanishes. Otherwise, one has the identity, [**59**], [**51**],

$$\text{(8.2)} \qquad \langle\, f_1 \otimes \bar{f}_2, \mathcal{E}_{2r}\big(\begin{pmatrix} \tau_1 & \\ & -\bar{\tau}_2 \end{pmatrix}, \bar{s}\,\big) \,\rangle_{\mathrm{Pet}} = \langle \sigma(\Phi_S(s)) f_1, f_2 \rangle_{\mathrm{Pet}}\, L^S(s + \frac{1}{2}, \pi)$$

where

$$\begin{aligned} S^{(r)}_{r+\frac{1}{2}} \ni f_i \longleftrightarrow F_i = {} & \text{ automorphic form } (i = 1,\, 2) \text{ for } H(\mathbb{A}), \text{ for } H = SO(r+1, r) \\ & \text{under the analogue of the Shimura–Waldspurger correspondence} \\ & \text{between forms of weight } r + \tfrac{1}{2} \text{ on } Mp_r \text{ and forms on } SO(r+1, r) \\ \sigma = {} & \text{ the irreducible automorphic cuspidal representation attached to } f_i \\ \pi = {} & \text{ the irreducible automorphic cuspidal representation attached to } F_i \\ L(s, \pi) = {} & \text{the degree } 2r \text{ Langlands L-function attached to } \pi, \\ & \text{and the standard representation of the L-group } H^\vee = Sp_r(\mathbb{C}) \\ \sigma(\Phi_S(s)) = {} & \text{ a local `twisting' operator which gives} \\ & \text{the contribution of bad local zeta integrals.} \end{aligned}$$

Considerations of local theta dichotomy, [**35**], [**40**], control the local root numbers of $L(s, \pi)$ so that the global root number is -1 and $L(\frac{1}{2}, \pi) = 0$. Combining (8.1) and (8.2), we obtain the **arithmetic inner product formula**:

$$\text{(8.3)} \qquad \langle\, \hat{\theta}_r(f), \hat{\theta}_r(f)\,\rangle \overset{??}{=} B \cdot \langle f, f \rangle_{\mathrm{Pet}}\, L'(\frac{1}{2}, \pi)$$

for a constant B coming the local zeta integrals at bad primes. Of course, this is only conjectural! For a general discussion of what one expects of such central critical values, cf. [**27**].

Examples.

1. The Gross-Kohnen-Zagier formula. In the case $n = r = 1$, we have

$$
\begin{aligned}
M &= \text{Shimura curve}\\
\mathcal{M} &= \text{integral model}\\
f &= \text{weight } \frac{3}{2} \qquad (8.4)\\
F &= \text{corresponding form of weight 2 (assumed a normalized newform)}\\
\pi &= \text{associated automorphic representation of } \mathrm{PGL}_2(\mathbb{A})\\
\hat{\theta}_1(f) &\in \widehat{\mathrm{CH}}^1(\mathcal{M})\\
L(s,\pi) &= L(s+\frac{1}{2}, F)\\
L(s,F) &= \text{ the standard Hecke L-function (with } s \mapsto 2-s \text{ functional eq.).}
\end{aligned}
$$

In this case, identity (8.3) becomes

$$\langle\, \hat{\theta}_1(f), \hat{\theta}_1(f)\,\rangle = B \cdot ||f||^2\, L'(1,F) \tag{8.5}$$

This is essentially the Gross-Kohnen-Zagier formula, [**29**], Theorem C.

2. Curves on Siegel 3-folds. The next example is $n = 3$ and $r = 2$. Then:

$$
\begin{aligned}
M &= \text{ Siegel 3-fold}\\
\mathcal{M} &= \text{ arithmetic 4-fold}\\
f &= \text{ a Siegel cusp form of weight } \frac{5}{2} \text{ and genus 2} \qquad (8.6)\\
\pi &= \text{ corresponding automorphic representation of } O(3,2)\\
L(s,\pi) &= \text{ the degree 4 L-function of } \pi
\end{aligned}
$$

The cycles $\mathcal{Z}(T)$ in the generating function $\widehat{\phi}_2(\tau)$ are now Shimura curves on the generic fiber $M = \mathcal{M}_{\mathbb{Q}}$, extended to arithmetic surfaces in the arithmetic 4-fold $\mathcal{M}$, and

$$\hat{\theta}_2(f) \in \widehat{\mathrm{CH}}^2(\mathcal{M}). \tag{8.7}$$

Then identity (8.3) says that the the central derivative of the degree 4 L-function $L(s,\pi)$ is expressible in terms of the height pairing $\langle\, \hat{\theta}_2(f), \hat{\theta}_2(f)\rangle$ of the class $\hat{\theta}_2(f)$ i.e., made out of the f–eigencomponents of 'curves on a Siegel 3-fold'. Of course, the proof of such a formula by the method outlined here requires that we prove the relevant versions of (5.7), (6.6) and (6.10), and, above all, the modularity of the codimension 2 generating function $\widehat{\phi}_2(\tau)$. Needless to say, this remains very speculative!

3. The central derivative of the triple product L-function. This case involves a slight variant of the previous pattern. If we take V of signature $(2,2)$, we have

$$M = \begin{cases} M_1 \times M_1, \quad M_1 = \text{modular curve or Shimura curve,} \\ \text{Hilbert–Blumenthal surface} \\ \text{compact Hilbert–Blumenthal type surface} \end{cases} \tag{8.8}$$

where, in the two cases in the first line, the discriminant of V is a square and $\text{witt}(V) = 2$ or 0, respectively, while, in the second two cases, $\boldsymbol{k} = \mathbb{Q}(\sqrt{\text{discr}(V)})$ is a real quadratic field and $\text{witt}(V) = 1$ or 0 respectively. Then, $\mathcal{M}$ is an arithmetic 3-fold, and, conjecturally, the generating function

$$\widehat{\phi}_1(\tau) \in \widehat{\text{CH}}^1(\mathcal{M}) \tag{8.9}$$

is a modular form of weight 2 for a subgroup $\Gamma' \subset \text{SL}_2(\mathbb{Z})$. Note that, on the generic fiber, the cycles $\mathcal{Z}(t)_{\mathbb{Q}}$ are the Hirzebruch–Zagier curves, [**52**]. For a cusp form $f \in S_2(\Gamma')$, we obtain a class

$$\hat{\theta}_1(f) \in \widehat{\text{CH}}^1(\mathcal{M}). \tag{8.10}$$

Consider the trilinear form on $\widehat{\text{CH}}^1(\mathcal{M})$ defined by

$$\langle\ \hat{z}_1, \hat{z}_2, \hat{z}_3\ \rangle := \widehat{\deg}\,(\hat{z}_1 \cdot \hat{z}_2 \cdot \hat{z}_3). \tag{8.11}$$

Then, for a triple of cusp forms of weight 2,

$$\begin{aligned} \langle\ \hat{\theta}(f_1), \hat{\theta}(f_2), \hat{\theta}(f_3)\ \rangle &= \langle\ f_1 f_2 f_3, \langle\ \widehat{\phi}_1(\tau_1), \widehat{\phi}_1(\tau_2), \widehat{\phi}_1(\tau_3)\ \rangle\ \rangle_{\text{Pet}} \\ &\overset{??}{=} \langle\ f_1 f_2 f_3, \widehat{\deg}\,(\widehat{\phi}_3(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}))\ \rangle_{\text{Pet}} \\ &\overset{??}{=} \langle\ f_1 f_2 f_3, \mathcal{E}_3'(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}, 0)\ \rangle_{\text{Pet}} \\ &= \frac{\partial}{\partial s}\Big\{\ \langle\ f_1 f_2 f_3, \mathcal{E}_3(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}, \bar{s})\ \rangle_{\text{Pet}} \Big\}\Big|_{s=0}. \end{aligned} \tag{8.12}$$

If we assume that the f_i's are newforms with associated cuspidal automorphic representations π_i, $i = 1, 2, 3$, then the integral in the last line is[11] the integral representation of the triple product L-function, [**23**], [**66**], [**30**], [**2**],

$$\langle\, f_1 f_2 f_3, \mathcal{E}_3(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}, \bar{s}\,)\,\rangle_{\text{Pet}} = B(s)\, L(s+\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi_3). \tag{8.13}$$

Note that the results of D. Prasad on dichotomy for local trilinear forms, [**67**], control the local root numbers and the 'target' space V.

Here, in addition to the modularity of the generating function $\widehat{\phi}_1(\tau)$, we have used the conjectural identities

$$\widehat{\phi}_1(\tau_1)\cdot\widehat{\phi}_1(\tau_2)\cdot\widehat{\phi}_1(\tau_3) \overset{??}{=} \widehat{\phi}_3(\begin{pmatrix} \tau_1 & & \\ & \tau_2 & \\ & & \tau_3 \end{pmatrix}), \tag{8.14}$$

analogous to (5.7) and

$$\widehat{\deg}\,\widehat{\phi}_3(\tau) \overset{??}{=} \mathcal{E}_3'(\tau, 0), \tag{8.15}$$

analogous to (6.6). The equality of certain coefficients on the two sides of (8.15) follows from the result of Gross and Keating, [**28**], and the formulas of Kitaoka, [**39**], cf. also [**52**].

In fact, one of the starting points of my long crusade to establish connections between heights and Fourier coefficients of central derivatives of Siegel Eisenstein series was an old joint project with Gross and Zagier, of which [**30**] was a preliminary 'exercise'. The other was my collaboration with Michael Harris on Jacquet's conjecture about the central value of the triple product L-function, [**33**], [**34**], based, in turn on a long collaboration with Steve Rallis on the Siegel–Weil formula. And, of course, the geometric picture which serves as an essential guide comes from joint work with John Millson. I would like to thank them all, together with my current collaborators Michael Rapoport and Tonghai Yang, for their generosity with their ideas, advice, encouragement, support and patience.

Appendix I: Shimura curves.

In this appendix, we illustrate some of our basic constructions in the case of modular and Shimura curves. In particular, this allows us to make a direct connection with classical Heegner points, one of the main themes of the MSRI conference.

[11] Almost, except that one must actually work with the similitude group GSp_3.

In the case of a rational quadratic space V of signature $(1,2)$, the varieties of part I are the classical Shimura curves. Let B be an indefinite quaternion algebra over $\mathbb{Q}$, and let

$$V = \{\ x \in B \mid \mathrm{tr}(x) = 0\ \}, \qquad Q(x) = \nu(x) = -x^2. \tag{A.I.1}$$

Note that the associated bilinear form is $(x,y) = \mathrm{tr}(xy^{\iota})$, where $x \mapsto x^{\iota}$ is the main involution on B. The action of $B^{\times}$ on V by conjugation induces an isomorphism

$$B^{\times} \xrightarrow{\sim} G = \mathrm{GSpin}(V). \tag{A.I.2}$$

We fix an isomorphism

$$B_{\mathbb{R}} = B \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} M_2(\mathbb{R}), \tag{A.I.3}$$

and obtain an identification

$$\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \xrightarrow{\sim} D, \qquad z \mapsto w(z) = \begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix} \quad \mathrm{mod}\ \mathbb{C}^{\times}. \tag{A.I.4}$$

Let S be the set of the primes p for which $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is a division algebra, and let $D(B) = \prod_{p \in S} p$. For a fixed maximal order O_B of B, there is an isomorphism

$$B(\mathbb{A}_f) \xrightarrow{\sim} \Big(\prod_{p\in S} B_p\Big) \times M_2(\mathbb{A}_f^S), \tag{A.I.5}$$

$$O_B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{\sim} \Big(\prod_{p\in S} O_{B,p}\Big) \times M_2(\widehat{\mathbb{Z}}^S).$$

For an integer N prime to $D(B)$, let R be the Eichler order of discriminant $ND(B)$ with

$$R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} \xrightarrow{\sim} \Big(\prod_{p\in S} O_{B,p}\Big) \times \{\ x \in M_2(\widehat{\mathbb{Z}}^S) \mid c \equiv 0 \ (\mathrm{mod}\ N)\ \}. \tag{A.I.6}$$

Then, for the compact open subgroup $K = (R \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^{\times} \subset G(\mathbb{A}_f)$, the quotient

$$X_0^B(N) := M_K(\mathbb{C}) \simeq G(\mathbb{Q})\backslash\Big(D \times G(\mathbb{A}_f)/K \Big) \simeq \Gamma \backslash D^+, \tag{A.I.7}$$

where $\Gamma = G(\mathbb{Q})^+ \cap K = R^{\times}$, is the analogue for B of the modular curve $X_0(N)$. Of course, when $B = M_2(\mathbb{Q})$, we need to add the cusps. The 0–cycles $Z(t,\varphi;K)$ are weighted combinations of CM–points. These can be described as follows. If we identify $V(\mathbb{Q})$ with a subset of $B_{\mathbb{R}} = M_2(\mathbb{R})$, then, for

$$x = \begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix} \in V(\mathbb{Q}) \subset M_2(\mathbb{R}), \qquad Q(x) = -(b^2 - 4ac), \tag{A.I.8}$$

$$D_x = \{\ z \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \ \mid\ (x, w(z)) = -2(az^2 + bz + c) = 0\ \}.$$

Note that for general B, the coordinates a, b and c of x need not lie in $\mathbb{Q}$.
For $d > 0$, let

$$\Omega_d = \{\ x \in V \mid Q(x) = d\ \} \tag{A.I.9}$$

and note that, if $x_0 \in \Omega_d(\mathbb{Q})$, then

$$\Omega_d(\mathbb{A}_f) = G(\mathbb{A}_f) \cdot x_0 = K \cdot \Omega_d(\mathbb{Q}). \tag{A.I.10}$$

By (iii) of Lemma 2.2 of [**41**], if $x \in V(\mathbb{Q})$, $g \in G(\mathbb{A}_f)$ and $\gamma \in G(\mathbb{Q})$, then, for the cycle defined by (2.8) above,

$$Z(\gamma x, \gamma g; K) = Z(x, g; K). \tag{A.I.11}$$

Thus, for any $\varphi \in S(V(\mathbb{A}_f))^K$, the weighted 0–cycle $Z(d, \varphi; K)$ on $X_0^B(N)$ is given by

$$Z(d, \varphi; K) = \sum_r \varphi(x_r)\, Z(x_r, 1; K), \tag{A.I.12}$$

with the notation of (2.11), where

$$\operatorname{supp}(\varphi) \cap \Omega_d(\mathbb{A}_f) = \coprod_r K \cdot x_r, \qquad x_r \in \Omega_d(\mathbb{Q}). \tag{A.I.13}$$

For example, if $L^\vee$ is the dual lattice of $L := R \cap V(\mathbb{Q})$, there is a Schwartz function

$$\varphi_\mu = \operatorname{char}(\mu + \widehat{L}) \in S(V(\mathbb{A}_f)) \tag{A.I.14}$$

for each coset $\mu + L$, for $\mu \in L^\vee$. Here $\widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$. The group Γ acts on $L^\vee/L$, and each Γ–orbit $\mathcal{O}$ defines a K–invariant weight function

$$\varphi_{\mathcal{O}} = \sum_{\mu \in \mathcal{O}} \varphi_\mu \quad \in S(V(\mathbb{A}_f))^K. \tag{A.I.15}$$

Then,

$$Z(d, \varphi_{\mathcal{O}}; K) = \sum_{\substack{x \in L^\vee \cap \Omega_d(\mathbb{Q}) \\ x + L \in \mathcal{O} \\ \text{mod } \Gamma}} \operatorname{pr}(D_x^+), \tag{A.I.16}$$

where $D_x^+ = D_x \cap D^+$ and $\operatorname{pr} : D^+ \to \Gamma\backslash D^+ = X_0^B(N)$ is the projection. Here each point $\operatorname{pr}(D_x^+)$ is to be counted with multiplicity e_x^{-1}, where $2e_x$ is the order of the stablizer of x in Γ.

The Heegner cycles studied by Gross, Kohnen, and Zagier, [**29**] can be recovered from this formalism in the case $B = M_2(\mathbb{Q})$. Of course, we take the standard identification $B_{\mathbb{R}} = M_2(\mathbb{R})$ and the maximal order $O_B = M_2(\mathbb{Z})$. For $x \in V$, we let

$$\text{(A.I.17)} \qquad y = \frac{1}{2}J^{-1}x = \frac{1}{2}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} b & 2c \\ -2a & -b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

This is the matrix for the quadratic form denoted by $[a, b, c]$ in [**29**], p.504. Moreover, if $g \in SL_2(\mathbb{Z})$, then the action of g on $[a, b, c]$ is given by $y \mapsto {}^t gyg$, and this amounts to

$$\text{(A.I.18)} \qquad x \mapsto g^{-1}xg$$

on the original x. Let

$$\text{(A.I.19)} \qquad L = \{x = \begin{pmatrix} b & c \\ -a & -b \end{pmatrix} \in M_2(\mathbb{Z}) \mid a \equiv b \equiv 0 \mod 2N, \text{ and } c \equiv 0 \mod 2\},$$

and, for a coset $r \in \mathbb{Z}/2N\mathbb{Z}$, let $\varphi_{N,r} \in S(V(\mathbb{A}_f))$ be the characteristic function of the set

$$\text{(A.I.20)} \qquad \begin{pmatrix} r & \\ & -r \end{pmatrix} + \hat{L} \quad \subset V(\mathbb{A}_f).$$

Note that the function $\varphi_{N,r}$ is K-invariant.

The set of x's which contribute to $Z(d, \varphi; K)$, i.e., the set $\mathrm{supp}(\varphi) \cap \Omega_d(\mathbb{Q})$, is

$$\text{(A.I.21)} \qquad \{\, x = \begin{pmatrix} b & 2c \\ -2a & b \end{pmatrix} \mid b^2 - 4ac = -d,\ a \equiv 0 \mod N,\ b \equiv r \mod 2N \,\}.$$

This set is mapped bijectively to the set
(A.I.22)

$$\mathcal{Q}_{N,r,d} := \{\, y = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \mid b^2 - 4ac = -d,\ a \equiv 0 \mod N \text{ and } b \equiv r \mod 2N \,\}$$

under the map $x \mapsto y$ of (A.I.17). Therefore $Z(d, \varphi; K)$ is precisely the image in $\Gamma_0(N)\backslash D^+$ of the set of roots z in D^+, identified with the upper half plane, of the quadratic equations $az^2 + bz + c = 0$, with $[a, b, c] \in \mathcal{Q}_{N,r,d}$. Note that, if

$$\text{(A.I.23)} \qquad \mathcal{Q}^+_{N,r,d} = \{y \in \mathcal{Q}_{N,r,d} \mid a > 0\},$$

then

$$\text{(A.I.24)} \qquad \mathcal{Q}_{N,r,d} \simeq \mathcal{Q}^+_{N,r,d} \cup \mathcal{Q}^+_{N,-r,d}.$$

The set of roots, counted with multiplicity, for $[a, b, c] \in \mathcal{Q}^+_{N,r,d}$ is denoted by $\mathcal{P}_{-d,r}$ in [**29**], p.542, and

$$\text{(A.I.25)} \qquad \mathcal{P}^*_{-d,r} = \mathcal{P}_{-d,r} \cup \mathcal{P}_{-d,-r},$$

where points are counted with the sum of their multiplicities in the two sets. We conclude that

Proposition A.I.1. *Fix N and r mod $2N$, and let $\varphi_{N,r}$ be as above. Then, for $K = K_0(N)$, as above,*

$$Z(d, \varphi_{N,r}; K) = \begin{cases} \mathcal{P}_{-d,r} + \mathcal{P}_{-d,-r} & \text{if } -d \equiv r^2 \bmod 4N \\ 0 & \text{otherwise,} \end{cases}$$

as 0–cycles on $X_0(N)$.

Appendix II: Theta forms.

This Appendix contains a few more details about the theta functions valued in differential forms (theta forms) used to prove Theorem 3.1, and about a formula for cup products which is analogous to the conjectural arithmetic inner product formula (8.3). Good references for the general construction of theta functions and automorphic forms via the Weil representation include [**73**], [**37**], [**59**]. Let $G' = \mathrm{Sp}(r)$ be the symplectic group of rank r over $\mathbb{Q}$, and let $G'_{\mathbb{A}}$ (resp. $G'_{\mathbb{R}}$) be the metaplectic cover[12] of $G'(\mathbb{A})$ (resp. $G'(\mathbb{R})$). Let $G'_{\mathbb{Q}} = G'(\mathbb{Q}) = \mathrm{Sp}_r(\mathbb{Q})$, identified with a subgroup of $G'_{\mathbb{A}}$ via the unique homomorphism $\mathrm{Sp}_r(\mathbb{Q}) = G'(\mathbb{Q}) \to G'_{\mathbb{A}}$ lifting the inclusion $G'(\mathbb{Q}) \hookrightarrow G'(\mathbb{A})$. Let ψ be the character of $\mathbb{A}$ which is trivial on $\mathbb{Q} \cdot \widehat{\mathbb{Z}}$ and with $\psi_\infty(x) = e(x) = e^{2\pi i x}$. Then $G'_{\mathbb{A}}$ acts on the space $S(V(\mathbb{A})^r)$ via the Weil representation $\omega = \omega_\psi$ determined by ψ, and this action commutes with the natural linear action of $H(\mathbb{A})$, where $H = O(V)$. By Poisson summation, the theta distribution Θ on $S(V(\mathbb{A})^r)$ defined by

$$\varphi \mapsto \Theta(\varphi) = \sum_{x \in V(\mathbb{Q})^r} \varphi(x) \tag{A.II.1}$$

is invariant under the action of $\gamma \in G'_{\mathbb{Q}}$, i.e.,

$$\Theta(\omega(\gamma)\varphi) = \Theta(\varphi). \tag{A.II.2}$$

The function

$$\theta(g', h; \varphi) := \sum_{x \in V(\mathbb{Q})^r} \omega(g')\varphi(h^{-1}x) \tag{A.II.3}$$

on $G'_{\mathbb{Q}} \backslash G'_{\mathbb{A}} \times H(\mathbb{Q}) \backslash H(\mathbb{A})$ is slowly increasing.

Returning to the Schwartz form $\varphi^r_\infty(\tau)$ of (3.2), we have the basic identity

$$\omega_\infty(g')\varphi^r_\infty(\tau) = j(g', \tau)^{-(n+2)}\, \varphi^r_\infty(g'(\tau)), \tag{A.II.4}$$

[12] The covers are only needed when n and hence $\dim(V)$ is odd. When n is even, one can simply work with the linear groups $G'(\mathbb{A}) = \mathrm{Sp}_r(\mathbb{A})$, $G'(\mathbb{R}) = \mathrm{Sp}_r(\mathbb{R})$, etc.

where $g' \in G'_{\mathbb{R}}$ acts on $S(V(\mathbb{R})^r)$ via the Weil representation determined by ψ_∞, and $j(g',\tau)$ is an automorphy factor with

$$j(g',\tau)^2 = \det(c\tau+d), \qquad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{A.II.5}$$

is the image of g' in $G'(\mathbb{R}) = \mathrm{Sp}_r(\mathbb{R})$. For $\varphi \in S(V(\mathbb{A}_f)^r)$, the theta function $\theta_r(\tau,\varphi)$ of (3.4) is given by

$$\theta_r(\tau,\varphi) = \Theta(\varphi^r_\infty(\tau)\otimes\varphi), \tag{A.II.6}$$

up to a translation by $h \in G(\mathbb{A}_f)$. Its transformation law is determined as follows. For $\gamma \in G'_{\mathbb{Q}}$, write $\gamma = \gamma_\infty\gamma_f$, with $\gamma_\infty \in G'_{\mathbb{R}}$ and $\gamma_f \in G'_{\mathbb{A}_f}$. Then, using (A.II.4), t

$$\begin{aligned} \theta_r(\gamma(\tau),\varphi) &= j(\gamma_\infty,\tau)^{(n+2)}\Theta(\omega_\infty(\gamma_\infty)\varphi^r_\infty(\tau)\otimes\varphi) \\ &= j(\gamma_\infty,\tau)^{(n+2)}\,\Theta(\omega(\gamma)\big(\varphi^r_\infty(\tau)\otimes\omega_f(\gamma_f)^{-1}\varphi\big)) \\ &= j(\gamma_\infty,\tau)^{(n+2)}\,\Theta(\varphi^r_\infty(\tau)\otimes\omega_f(\gamma_f)^{-1}\varphi) \\ &= j(\gamma_\infty,\tau)^{(n+2)}\,\theta_r(\tau,\omega_f(\gamma_f)^{-1}\varphi). \end{aligned} \tag{A.II.7}$$

This is valid for any $\varphi \in S(V(\mathbb{A}_f)^r)$ and any $\gamma \in G'_{\mathbb{Q}}$. To obtain a more traditional transformation law, let $K' \subset G'_{\mathbb{A}_f}$ be the inverse image of $\mathrm{Sp}_r(\widehat{\mathbb{Z}})$, and let Γ' be the inverse image of $\mathrm{Sp}_r(\mathbb{Z})$ in $G'_{\mathbb{R}}$. Note that, for any $\gamma' \in \Gamma'$, there is a unique element $k' \in K'$ such that $\gamma' k' = \gamma \in G'_{\mathbb{Q}}$. Suppose that $L \subset V$ is a lattice on which the quadratic form Q is $\mathbb{Z}$–valued, and let $L^\vee$ be the dual lattice. Let $\widehat{L} = L\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$ and $\widehat{L}^\vee = L^\vee\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$. Let $S_L \subset S(V(\mathbb{A}_f)^r)$ be the subspace of functions φ such that

$$\mathrm{supp}(\varphi) \subset (\widehat{L}^\vee)^r \tag{A.II.8}$$

and φ is constant on cosets of $\widehat{L}^r$. Then S_L is preserved under the Weil representation action ω_f of K'. Define a representation ρ_L of Γ' on S_L by

$$\rho_L(\gamma') = \omega_f(k'), \tag{A.II.9}$$

and let $\rho_L^\vee$ be the associated representation on the dual space $S_L^\vee$. The map

$$\theta_r(\tau):\ \varphi \mapsto \theta_r(\tau,\varphi), \tag{A.II.10}$$

defines an element of $S_L^\vee$, and the transformation law (A.II.7) amounts to the classical transformation law

$$\theta_r(\gamma(\tau)) = j(\gamma',\tau)^{n+2}\,\rho_L^\vee(\gamma')\theta_r(\tau), \qquad \gamma \in \mathrm{Sp}_r(\mathbb{Z}) \tag{A.II.11}$$

for the vector valued form $\theta_r(\tau)$, of type $(\rho_L^\vee, S_L^\vee)$ in the style of Borcherds, [**4**], [**44**].

Finally, we mention the analogue in the geometric situation of the conjectural arithmetic inner product formula (8.3). This formula is a geometric version of the Rallis inner product formula, [**68**], [**51**]. A more general version of such a formula was used by Jian-Shu Li, [**59**], to obtain nonvanishing results for the cohomology of locally symmetric spaces attached to classical groups. Here we only consider the inner product on the middle dimensional cohomology.

Suppose that $n = 2r$ and that the Shimura variety M_K is compact. To obtain a large supply of such examples, one can work over a totally real field, as explained at the end of section 1. For a Siegel cusp form f of genus r and weight $\frac{n}{2}+1$, define the classical theta lift by

$$\text{(A.II.12)}\qquad \begin{aligned}\theta_r(f,\varphi) &:= \langle\, f, \theta_r(\cdot,\varphi)\,\rangle_{\mathrm{Pet}}\\ &= \int_{\Gamma'\backslash\mathfrak{H}_r} f(\tau)\,\overline{\theta_r(\tau,\varphi)}\,\det(v)^{\frac{n}{2}+1}\,d\mu(\tau).\end{aligned}$$

Here $\langle\ ,\ \rangle_{\mathrm{Pet}}$ denotes the Petersson inner product. Thus, we get a map,

$$\text{(A.II.13)}\qquad S^{(r)}_{\frac{n}{2}+1} \longrightarrow \mathrm{H}^{2r}(M), \qquad f \mapsto [\,\theta_r(f,\varphi)\,],$$

whose image lies in the subspace spanned by the cohomology classes of the cycles $Z(T,\varphi)$. If $\varphi \in S(V(\mathbb{A}_f)^r)^K$ is K–invariant, then $[\,\theta_r(f,\varphi)\,] \in \mathrm{H}^{2r}(M_K)$. Consider the pairing of two such classes:

$$\text{(A.II.14)}\qquad \begin{aligned}(\,[\,\theta_r(f_1,\varphi_1)\,],[\,\theta_r(f_2,\varphi_2)\,]\,) &= \int_{M_K} \theta_r(f_1,\varphi_1)\wedge\overline{\theta_r(f_2,\varphi_2)}\\ &= \langle\ f_1\otimes\bar f_2, \int_{M_K}\theta_r(\tau_1,\varphi_1)\wedge\overline{\theta_r(\tau_2,\varphi_2)}\ \rangle_{\mathrm{Pet}}\\ &= \langle\ f_1\otimes\bar f_2, \int_{M_K}\theta_n(\begin{pmatrix}\tau_1 & \\ & -\bar\tau_2\end{pmatrix}, \varphi_1\otimes\bar\varphi_2)\ \rangle_{\mathrm{Pet}}\\ &= \mathrm{vol}(M_K,\Omega^n)\cdot\langle\ f_1\otimes\bar f_2, E_{2n}(\begin{pmatrix}\tau_1 & \\ & -\bar\tau_2\end{pmatrix}, \frac{1}{2}, \varphi_1\otimes\bar\varphi_2)\ \rangle_{\mathrm{Pet}}\end{aligned}$$

by (3.7) and (4.4). In contrast to the situation in (8.1) and (8.2), the doubling integral occuring in the last line here involves Siegel modular cusp forms $f_i \in S^{(r)}_{r+1}$ of *integral* weight. The integral is zero unless the f_i's correspond to the same irreducible cuspidal automorphic representation π of $\mathrm{Sp}_r(\mathbb{A})$, in which case the doubling integral gives, [**51**], (7.2.8), p.69,

(A.II.15)

$$\langle\ f_1\otimes\bar f_2, E_{2n}(\begin{pmatrix}\tau_1 & \\ & -\bar\tau_2\end{pmatrix}, \bar s, \varphi_1\otimes\bar\varphi_2)\ \rangle_{\mathrm{Pet}} = \frac{1}{b^S_{2r}(s,\chi_V)}\cdot\langle\pi(\Phi_S(s))f_1, f_2\rangle_{\mathrm{Pet}}\, L^S(s+\frac{1}{2},\pi)$$

where

(A.II.16)

$$L(s,\pi) = \text{the degree } 2r+1 \text{ Langlands L-function attached to } \pi,$$
$$\text{and the standard representation of the L-group } \mathrm{Sp}_r^\vee = SO(2r+1,\mathbb{C})$$

$$\pi_S(\Phi(s)) = \text{a convolution operator, determined by } \varphi_1\otimes\bar\varphi_2, \text{ which gives}$$
$$\text{the contribution of bad local zeta integrals.}$$

$$b_{2r}(s,\chi_V) = L(s+r+\frac{1}{2},\chi_V)\cdot\prod_{k=1}^{r}\zeta(2s+2k-1),$$

and the superscript S indicates that the Euler factors for places in a finite set S (including the archimedean places) have been omitted. We refer the reader to sections 7 and 8 of [**51**] for more details. Note that, in the arithmetic situation of section 8, the factor analogous to $b_{2r}(s,\chi_V)$ is included in the normalization of the Eisenstein series $\mathcal{E}_{2r}(\tau,s)$ and hence does not show up in (8.2). Thus, we obtain the analogue of (8.3) for the cup product:

(A.II.17)
$$(\,[\,\theta_r(f,\varphi)\,],[\,\theta_r(f,\varphi)\,]\,) = B\cdot\langle f,f\rangle_{\mathrm{Pet}}\,L(1,\pi),$$

where we have lumped various constant factors in B.

We also note that, in our anisotropic case, Theorem 4.1 gives

(A.II.18)
$$\mathrm{vol}(M_K,\Omega^n)\cdot E_n(\tau,\frac{1}{2},\varphi) = \sum_{T\geq 0}\mathrm{vol}(Z(T,\varphi))\,q^T$$

and, for $r_1+r_2=n$, the analogue of the conjectural identity (6.10) is the formula

(A.II.19)
$$([\theta_{r_1}(\tau_1,\varphi_1)],[\theta_{r_2}(\tau_2,\varphi_2)]) = \mathrm{vol}(M_K,\Omega^n)\cdot E_n(\begin{pmatrix}\tau_1 & \\ & -\bar\tau_2\end{pmatrix},\frac{1}{2},\varphi_1\otimes\bar\varphi_2),$$

expressing the cup product of the cohomology generating functions valued in $\mathrm{H}^{2r_1}(M_K)$ and $\mathrm{H}^{2r_2}(M_K)$ as the pullback to $\mathfrak{H}_{r_1}\times\mathfrak{H}_{r_2}$ of this Eisenstein series.

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