# Welcome back to MAT137!

(Section L5101, Tuesday 6-9 and Thursday 6-9)

- PS3 is due on Friday.
- **For next day's lecture, watch videos 6.1, 6.2, 6.4, 6.5, 6.7, 6.8, 6.10.**
- **Today's topics: Rolle's Theorem, Mean Value Theorem, Monotonicity** (Videos 5.5 - 5.12)

# Let's get started!!

Any question from previous class?

# Fractional exponents

Let $g(x) = x^{2/3}(x-1)^3$.

Find local and global extrema of $g$ on $[-1, 2]$.

## What can you conclude?

We know the following about the function $f$.

- $f$ has domain $\mathbb{R}$.
- $f$ is continuous
- $f(0) = 0$
- For every $x \in \mathbb{R}$, $f(x) \geq x$.

What can you conclude about $f'(0)$? Prove it.

*Hint:* Consider the function $g(x) = f(x) - x$. How does $g$ behave at 0?

# Zeroes of functions and their derivatives

For each part, construct a function $f$ that is differentiable on $\mathbb{R}$ and such that:

1. $f$ has exactly 2 zeroes and $f'$ has exactly 1 zero.
2. $f$ has exactly 2 zeroes and $f'$ has exactly 2 zeroes.
3. $f$ has exactly 3 zeroes and $f'$ has exactly 1 zero.
4. $f$ has exactly 1 zero and $f'$ has infinitely many zeroes.

(A sketch of a graph is good enough for each part.)

# Finding the number of roots of a function

**Problem.** Let

$$f(x) = e^x - \sin x + x^2 + 10x.$$

How many zeroes does $f$ have?

# A nice consequence of Rolle's Theorem

**Theorem**

*Let $a < b$ be real numbers. Let $f$ be a function defined on $[a, b]$.*

*IF*

- *$f$ is continuous on $[a, b]$ and differentiable on $(a, b)$*
- $\forall x \in (a, b), f'(x) \neq 0$.

*THEN $f$ is one-to-one on $[a, b]$.*

1. Write the definition of "$f$ is not injective on $[a, b]$". You will need it.
2. Recall the statement of Rolle's Theorem. You will need that too.
3. Do some rough work to understand why this is true.
4. Write the proof.

# Roots of a polynomial

Let $n \in \mathbb{N}$.
A **polynomial** of degree $n$ is a function $P(x) = a_0 + a_1 x + a_2 x^2 + \dots a_n x^n$, where $a_i \in \mathbb{R}$ for all $i = 0, 1, 2, \dots, n$ and $a_n \neq 0$.

Prove that a polynomial of degree $n$ has at most $n$ zeroes/roots.

# Warm up

- Let $f$ be a function defined on an interval $I$. Write the definition of "$f$ is increasing on $I$".

- Write the statement of the Mean Value Theorem.

# Positive derivative implies increasing in an open interval

Use the MVT to prove

**Theorem**

Let $a < b$. Let $f$ be a differentiable function on $(a, b)$.

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN $f$ is increasing on $(a, b)$.

1. Recall the definition of what you are trying to prove.
2. From that definition, figure out the structure of the proof.
3. If you have used a theorem, did you verify the hypotheses?

# Is this proof correct?

## Theorem

Let $a < b$. Let $f$ be a differentiable function on $(a, b)$.

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN $f$ is increasing on $(a, b)$.

## Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$, so $\quad f(b) > f(a)$
- $f$ is increasing.

# Positive derivative implies increasing on a closed interval

What conditions would you need here?

**Theorem**

Let $a < b$ and $f$ be differentiable on ..., continuous on ...,

- IF $\forall x \in$ ..., $f'(x) > 0$,
- THEN $f$ is increasing on $[a, b]$.

# Intervals of monotonicity

Let $g(x) = x^3(x^2 - 4)^{1/3}$.

Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.

To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2 - 36)}{3(x^2 - 4)^{2/3}}$$

# Cauchy's MVT - Part 1 (If time permits)

Here is a new theorem:

**We want to prove this Theorem**

Let $a < b$. Let $f$ and $g$ be functions defined on $[a, b]$.
IF (some conditions)
THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

**What is wrong with this "proof"?**

- By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- By MVT, $\exists c \in (a, b)$ s.t. $g'(c) = \dfrac{g(b) - g(a)}{b - a}$
- Divide the two equations and we get what we wanted.

$\square$

# Cauchy's MVT - Part 2 (If time permits)

**We want to prove this theorem**

Let $a < b$. Let $f$ and $g$ be functions defined on $[a, b]$.
IF

- $f$ and $g$ are continuous on $[a, b]$,
- $f$ and $g$ are differentiable on $(a, b)$,
- $g(b) \neq g(a)$

THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

1. There is one number $M \in \mathbb{R}$ so that you will be able to apply Rolle's Theorem to the new function $H(x) = f(x) - Mg(x)$ on the interval $[a, b]$. What is $M$?
2. Apply Rolle's Theorem to $H$. What do you conclude?
3. Fill in the missing hypotheses in the theorem above.
4. Prove it.

# Inequalities

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

*Hint:* When is the function $f(x) = e^x - 1 - x$ increasing or decreasing?

# Proving difficult identities

Prove that, for every $x \geq 0$,

$$\arcsin \frac{1-x}{1+x} + 2 \arctan \sqrt{x} = \frac{\pi}{2}$$

*Hint:* Take derivatives.