Welcome back to MAT137- Section L5101

- **Before next class:**
  - **Watch videos 2.10, 2.11**

Let's get started!!

Today's videos: 2.7, 2.8, 2.9

Today's topic: The formal definition of limit

Any question from previous class?

Infinite limits - 2

Which one(s) is the definition of $\lim_{x \to a} f(x) = \infty$?

1. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
2. $\forall M \in \mathbb{Z}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
3. $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
4. $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
5. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) \geq M$
6. $\forall M < 10, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) \geq M$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > 100$$

1. Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > M$$

Related implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > 100$$

1. Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \quad \Longrightarrow \quad f(x) > M$$

2. Which values of $\delta > 0$ satisfy ... ?

$$0 < |x - a| < \delta \quad \Longrightarrow \quad f(x) > 100$$

More negation

Let f be a function with domain $\mathbb{R}$. Write the negation of the statement:

$$\text{IF } \; 2 < x < 4, \qquad \text{THEN } \; 1 < f(x) < 3.$$

Existence

Write down the formal definition of the following statements:

1. $\lim_{x \to a} f(x) = L$

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x)$ does not exist

Implications

Suppose you know:

1. If $|x - a| < 2$, then A is true.
2. If $|x - a| < 5$, then B is true.

What condition do you need to guarantee both A and B are true?

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Suppose you know:

1. If $|x - a| < 2$, then A is true.
2. If $|x - a| < 5$, then B is true.

What condition do you need to guarantee both A and B are true?

Suppose you know:

1. If $x > 100$, then A is true.
2. If $x > 1000$, then B is true.

What condition do you need to guarantee both A and B are true?

Preparation: choosing deltas

1. Find a value of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
2. Find *all* values of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 1.$$
3. Find a value of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$
4. Let us fix $\varepsilon > 0$. Find a value of $\delta > 0$ such that
$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

What is wrong with this “proof”?

Prove that
$$\lim_{x\to 3}(5x+1)=16$$

“Proof:”

Let $\varepsilon > 0$.
WTS $\forall\varepsilon > 0$, $\exists\delta > 0$ s.t.
$$0<|x-3|<\delta \implies |(5x+1)-(16)|<\varepsilon$$

$$|(5x+1)-(16)|<\varepsilon \iff |5x+15|<\varepsilon$$
$$\iff 5|x+3|<\varepsilon \implies \delta=\frac{\varepsilon}{3}$$

□

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

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2. Write down what the structure of the formal proof should be, without filling the details.

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

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Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (2).

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (2).
2. Write down what the structure of the formal proof should be, without filling the details.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (2).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \qquad (2)$$

directly from the definition.

1. Write down the formal definition of the statement (2).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. Write down a complete formal proof.

Is this proof correct?

Claim:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$$

Proof:

- Let $\varepsilon > 0$.
- Take $\delta = \sqrt{\dfrac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3+x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$. □