Welcome back to MAT137!

(Section L5101, Tuesday 6-9 and Thursday 6-9)

- Tutorials started this week.
- **For next day's lecture, watch videos 2.14 through 2.20.**
- **Today's topics: Formal Proofs With Limits, Limit Laws, The Squeeze Theorem, and More.** (Videos 2.7–2.13)

How to work?

- Open today's slides alongside Zoom.
- Take notes and solve problems like in-person class.
- **Mute your mic and camera to avoid lag**. Please, without exception!
- Reply to Polls!
- Use the chat if you have any question/doubt and when you give an answer
- Discuss in Breakout rooms. Socialize with your peers in the rooms, discuss math (use the audio/video).
- Ask questions, answer them (using the chat only!). Don't be afraid/shy to ask ("silly"!!) questions or be wrong in class.

Let's get started!!

Any question from previous class?

Topics: Formal Proofs With Limits, Limit Laws, The Squeeze Theorem, and More

Infinite limits

Definition

Let $a, L \in \mathbb{R}$.

Let f be a function defined at least on an interval around a, except possibly at a.

Write formal definitions for

1. $\lim_{x \to a} f(x) = \infty$.
2. $\lim_{x \to \infty} f(x) = L$.

Hint: What does it mean mathematically to say something $\to \infty$, that is, it becomes arbitrarily large?

Infinite limits

Which statement(s) below are the definition of

$$\lim_{x \to a} f(x) = \infty$$

1. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
2. $\forall M > 0, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
3. $\forall M > 5, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
4. $\forall M \in \mathbb{N}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) > M$
5. $\forall M \in \mathbb{R}, \exists \delta > 0$ s.t. $0 < |x - a| < \delta \implies f(x) \geq M$

Existence

Write down the formal definition of the following statements:

1. $\lim\limits_{x \to a} f(x)$ exists
2. $\lim\limits_{x \to a} f(x)$ does not exist

Implications

Suppose you know:

1. If $|x - a| < 2$, then A is true.
2. If $|x - a| < 5$, then B is true.

What condition do you need to guarantee both A and B are true?

Suppose you know:

1. If $x > 100$, then A is true.
2. If $x > 1000$, then B is true.

What condition do you need to guarantee both A and B are true?

Preparation: choosing deltas

1. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

2. Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

3. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

4. Let us fix $\varepsilon > 0$. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

A harder $\epsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 1} x^2 = 1 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the claim
2. Write down what the structure of the formal proof should be, without filling the details.
3. **Rough work: What is** δ?
4. Write down a complete formal proof.

A harder $\epsilon - \delta$ proof: Rough work

Goal

We want to prove that

$$\lim_{x \to 1} x^2 = 1 \tag{3}$$

directly from the definition.

1. Write down the formal definition of the claim
2. Start with the $|f(x) - L|$ part of the definition. Algebraically manipulate it to factorize to get several terms
3. Determine which one of the terms you can make arbitrarily small by constraining $|x - a|$
4. Bound all other terms by (reasonable) constants by constraining $|x - a|$.
5. Now choose δ.

Is this proof correct?

Claim:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\quad 0 < |x-1| < \delta \implies |x^2-1| < \varepsilon.$

Proof.

- Let $\varepsilon > 0$. Take $\delta = \dfrac{\varepsilon}{|x+1|}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x-1| < \delta$. Then

$$|x^2-1| = |x-1||x+1| < \delta|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^2-1| < \varepsilon$.

$\square$

A theorem about limits

Theorem

Let f be a function with domain $\mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = 3.$$

Prove that

$$\lim_{x \to 0} \left[5\, f(2x)\right] = 15$$

directly from the definition of the limit. Do not use any limit laws.

1. Write down the formal definition of what you want to prove.
2. Write down what the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
   Do you start from what you know and prove what you have to prove?
6. Did you remember not to assume your conclusion?

Squeeze to infinity

Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x) \leq h(x)$
- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} h(x) = L$

THEN $\lim_{x \to a} g(x) = L$.

Come up with a new version of this theorem which instead concludes that

$$\lim_{x \to a} g(x) = \infty$$

Hint: Draw a picture of the Squeeze Theorem. Then draw a picture of the new theorem.

Squeeze to infinity

Squeeze Theorem (new version)

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x)$
- $\lim_{x \to a} f(x) = \infty$

THEN $\lim_{x \to a} g(x) = \infty$.

1. Write down the definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work
4. Write down a complete, formal proof.

True or false?

Is this claim true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$

A new theorem about products

Theorem

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is **bounded**. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
6. Did you only use things you know to be true?
   Did you prove what you have to prove?
7. Did you remember not to assume your conclusion?

Product rule

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = L$, and
- $\lim_{x \to a} g(x) = M$.

Show that g is bounded near a, that is

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

Hence show that $\lim_{x \to a} [f(x)g(x)] = LM$
(Hint: Write $f(x) = (f(x) - L) + L$ and use the previous theorem)