Welcome back to MAT137!

(Section L5101, Tuesday 6-9 and Thursday 6-9)

- Problem set 2 due this Friday
- My office hour is tomorrow 5:30-7
- **For next day's lecture, watch videos 2.21, 2.22, 3.1-3.5, 3.8.**
- **Today's topics: Continuity and Limit Computations** (Videos 2.14–2.20)

How to work?

- Open today's slides alongside Zoom.
- **Mute your mic and camera to avoid lag**. Please, without exception!
- Reply to Polls!
- Use the chat if you have any question/doubt and when you give an answer
- Ask questions.

Let's get started!!

Any question from previous class?

What does it mean to say that a function f is continuous at a?

Squeeze to infinity

Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x) \leq h(x)$
- $\lim\limits_{x \to a} f(x) = L$
- $\lim\limits_{x \to a} h(x) = L$

THEN $\lim\limits_{x \to a} g(x) = L$.

Come up with a new version of this theorem which instead concludes that

$$\lim_{x \to a} g(x) = \infty$$

Hint: Draw a picture of the Squeeze Theorem. Then draw a picture of the new theorem.

Squeeze to infinity (Exc.)

Squeeze Theorem (new version)

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x)$
- $\lim\limits_{x \to a} f(x) = \infty$

THEN $\lim\limits_{x \to a} g(x) = \infty$.

1. Write down the definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work
4. Write down a complete, formal proof.

True or false?

Is this claim true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$

A new theorem about products

Theorem

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is **bounded**. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
6. Did you only use things you know to be true?
   Did you prove what you have to prove?
7. Did you remember not to assume your conclusion?

Product rule: How to prove it?

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = L$, and
- $\lim_{x \to a} g(x) = K$.

Show that g is bounded near a, that is

$$\exists M > 0, p > 0 \text{ s.t. } 0 < |x - a| < p \Rightarrow |g(x)| \leq M.$$

Hence show that $\lim_{x \to a} [f(x)g(x)] = LK$
(Hint: Write $f(x) = (f(x) - L) + L$ and use the theorem in slide $\#7$)

How to write limits at infinity

Goal

We want to prove that

$$\lim_{x \to \infty} \frac{x+1}{x} = 1 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement.
2. Write down what the structure of the formal proof should be, without filling the details.
3. **Rough work. What is M?**
4. Write down a complete formal proof.

True or false?

Claim

Let $a, L \in \mathbb{R}$. IF f and g are functions such that

A. $\lim_{x \to a} f(x) = L$
B. $\lim_{y \to L} g(y) = M$

THEN

$$\lim_{x \to a} g(f(x)) = \lim_{y \to L} g(y) = M$$

This is FALSE.

Two ways to fix it (Proofs Exc.)

FALSE claim

Let $a, L \in \mathbb{R}$. IF f and g are functions such that

- (A) $\lim_{x\to a} f(x) = L$
- (B) $\lim_{y\to L} g(y) = M$

THEN

$$\lim_{x\to a} g(f(x)) = \lim_{y\to L} g(y) = M$$

Recall $\lim_{x\to a} f(x) = L$ means

$$\left\{ x \text{ close to } a \quad \text{and} \quad x \neq a \right\} \implies f(x) \text{ close to } L$$

And $\lim_{y\to L} g(y) = M$ means, replacing y by $f(x)$,

$$\left\{ f(x) \text{ close to } L \quad \text{and} \quad f(x) \neq L \right\} \implies g(f(x)) \text{ close to } M$$

1. Replace (B) with a condition to fix the theorem.
2. Alternatively, can you replace (A) with a condition to fix the theorem?

Difficult examples

Construct a function f such that

- $\lim_{x \to 0} f(x) = 0$
- $\lim_{x \to 0} f(f(x)) = 1$

Construct a pair of functions f and g such that

- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 1} g(x) = 2$
- $\lim_{x \to 0} g(f(x)) = 42$

A question from an old test

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to determine the following limits:

1. $\lim_{x \to 0} \dfrac{g(x)}{x}$
2. $\lim_{x \to 0} \dfrac{g(x)}{x^4}$
3. $\lim_{x \to 0} \dfrac{g(3x)}{x^2}$

Computations!

Using that $\lim_{x \to 0} \frac{\sin x}{x} = 1$, compute the following limits:

1. $\lim_{x \to 2} \frac{\sin x}{x}$
2. $\lim_{x \to 0} \frac{\sin(5x)}{x}$
3. $\lim_{x \to 0} \frac{\tan^2(2x^2)}{x^4}$
4. $\lim_{x \to 0} \frac{\sin e^x}{e^x}$
5. $\lim_{x \to 0} \frac{1 - \cos x}{x}$
6. $\lim_{x \to 0} \frac{\tan^{10}(2x^{20})}{\sin^{200}(3x)}$

Limits at infinity

Compute:

1. $\lim\limits_{x \to \infty} \left(x^7 - 2x^5 + 11\right)$
2. $\lim\limits_{x \to \infty} \left(x^2 - \sqrt{x^5 + 1}\right)$
3. $\lim\limits_{x \to \infty} \dfrac{x^2 + 11}{x + 1}$
4. $\lim\limits_{x \to \infty} \dfrac{x^2 + 2x + 3}{3x^2 + 4x + 5}$
5. $\lim\limits_{x \to \infty} \dfrac{x^3 + \sqrt{2x^6 + 1}}{2x^3 + \sqrt{x^5 + 1}}$

Rational limits

Compute:

1. $$\lim_{x \to -3^+} \frac{x^2 - 9}{3 - 2x - x^2}$$
2. $$\lim_{x \to 1^+} \frac{x^2 - 9}{3 - 2x - x^2}$$