Welcome back to MAT137!

(Section L5101, Tuesday 6-9 and Thursday 6-9)

- Quiz 2 at the end of today's class (from 8:40-9 pm)
- **For next day's lecture, watch videos 4.3-4.8, 5.2-5.4.**
- **Today's topics: Derivatives of Exponentials and Logarithms, Related Rates, Inverse Functions** (Videos 3.13, 3.15-3.20, 4.1, 4.2)

Let's get started!!

Any question from previous class?

Explicit functions

An equation like $y = x^2 + \sin(x)$ expresses a relationship between values of x and y.

More specifically, it says that the values of y that satisfy the equation are related to the values of x by a **function**.

(The function is $f(x) = x^2 + \sin(x)$.)

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos(x).$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y. The set of all points (x, y) that satisfy the above equation is the unit circle in the plane.

In this case though, the values of y **can not** be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates all the information in the earlier equation.

But we still might want to ask how y varies when x varies.

Implicit functions

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases—when y is non-negative or non-positive—the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{y} \right)$.
When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{-\sqrt{1 - x^2}} = \dfrac{-x}{y} \right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$. This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$, as you saw in video 3.12.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\,y'.$$

In this case you'd get:

$$2x + 2y\,y' = 0 \implies y' = -\frac{x}{y}.$$

Notice that the RHS of this formula doesn't make sense when $y = 0$. That makes sense, since y *cannot be thought of as a function of* x around those points. What happens at $(1, 0)$ and $(-1, 0)$?

Implicit differentiation

The equation

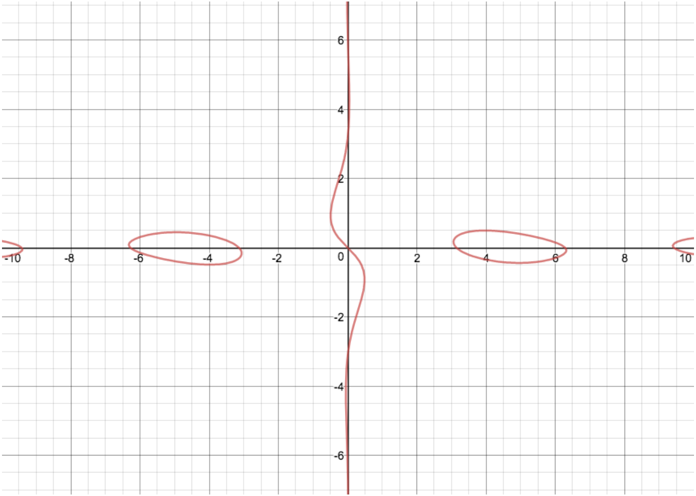
$$\sin(x+y) + xy^2 = 0$$

graph

defines a function $y = h(x)$ near $(0, 0)$.

Using implicit differentiation, compute

1. $h(0)$
2. $h'(0)$
3. $h''(0)$



6
4
2
0
-2
-4
-6
-10
-8
-6
-4
-2
2
4
6
8
10

A pesky function

Let $h(x) = x^2 \sin \frac{1}{x}$.

1. Calculate $h'(x)$ for any $x \neq 0$.
2. Using the definition of derivative, calculate $h'(0)$.
3. Is h continuous at 0?
4. Is h differentiable at 0?
5. Is h' continuous at 0?

Hint: The last two questions have different answers.

Some quick derivatives with exponentials and logarithms

Problem. Compute the derivatives of the following functions:

1. $f(x) = e^{\sin x + \cos x} \log(x)$
2. $f(x) = \pi^{\tan x}$
3. $f(x) = \ln\left[e^x + \ln(\ln(\ln(x)))\right]$

Reminder: We know:

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$
- $\dfrac{d}{dx} a^x = a^x \ln a$

Logarithmic differentiation.

Problem. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.

2. $g(x) = \sin x^{\cos x} + \cos x^{\sin x}$.

A different type of logarithm

Problem. Compute the derivative of

$$f(x) = \log_{x+1}\left(x^2 + 1\right).$$

Hint: If you don't know where to start, remember the definition of the logarithm:

$$\log_a b = c \iff a^c = b.$$

Congratulations!! You have now achieved full differentiation power!

With the tools you now know, you can more or less differentiate any function you can right down.

For example, you can compute the derivative of:

$$h(x) = \sqrt[3]{\frac{(\sin^6 x)\sqrt{x^7 + 6x + 2}}{3^x \left(x^{10} + 2x\right)^{10}}}$$

It will be long, but easy. Taking a log of both sides will turn the right side into a long sum, which is easy to differentiate.

Sliding ladder

A ten-meter long ladder is leaning against a vertical wall and sliding. The top end of the ladder is 8 meters high and sliding down at a rate of 1 meter per second.

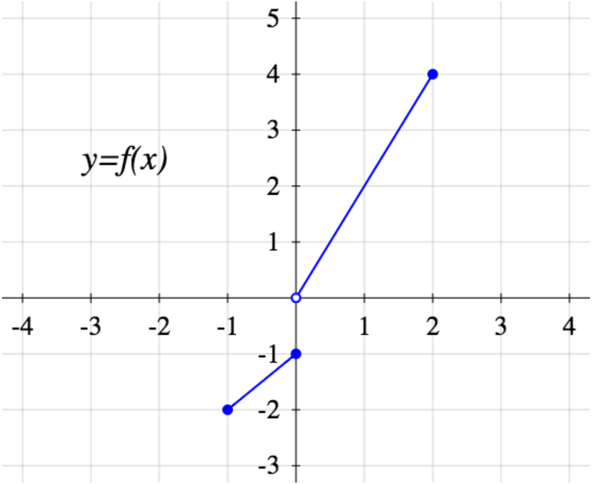
At what rate is the bottom end sliding?

Math party

The MAT137 TAs wanted to rent a disco ball for their upcoming party. However, they could only afford a flashlight. At the party, one TA is designated the "human disco ball". The TA stands in the center of the room pointing the flashlight horizontally and spins at 3 revolutions per second. The room is square with side length 8 meters.

At what speed is the light from the flashlight moving across the wall when it is 2 meters away from a corner?

Inverse function from a graph



Calculate:

1. $f(2)$
2. $f(0)$
3. $f^{-1}(2)$
4. $f^{-1}(0)$
5. $f^{-1}(-1)$

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

1. Calculate $h^{-1}(-8)$.
2. Find an equation for $h^{-1}(x)$.
3. Sketch the graphs of h and h^{-1}.
4. Verify that for every $x \in \boxed{???}$, $h(h^{-1}(x)) = x$
5. Verify that for every $x \in \boxed{???}$, $h^{-1}(h(x)) = x$.