MAT137

(Section L0501, September 25, 2019)

Instructor	Sourav Sarkar
Email	`ssarkar@math.toronto.edu` (**always** put "MAT137" in the subject line)
Office hours	**Thursdays 4-6, PG003**

Don't forget!

- Course website: http://uoft.me/MAT137
- Webpage for this section: http://www.math.toronto.edu/ssarkar/137.html
- **For next day's lecture, watch videos 2.10 and 2.11**
- Today's lecture will **assume** you have watched videos 2.5, 2.6, 2.7, 2.8 (and 2.9).

Let's get started!!

Topics: Definition of limit and proofs from the definition

Warm-ups and Reviews

1. Factorize $x^2 - a^2$ and $x^3 - a^3$.

2. Find *all* positive values of B, and C that make the following implication true

$$|x - 3| < 1 \implies B < |x + 5| < C$$

Floor

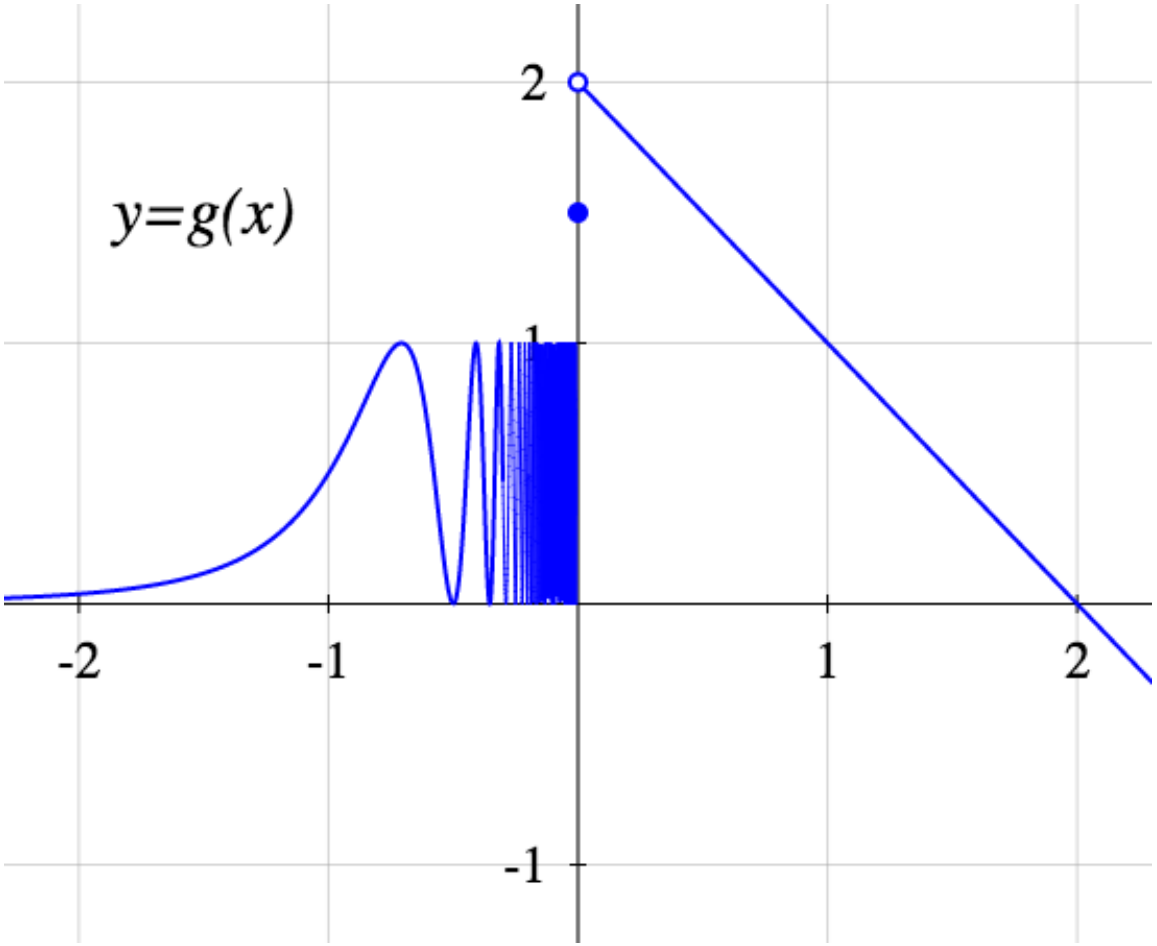
Given a real number x, we defined the **floor of** x, denoted by $\lfloor x \rfloor$, as the largest integer smaller than or equal to x. For example:

$$\lfloor \pi \rfloor = 3, \qquad \lfloor 7 \rfloor = 7, \qquad \lfloor -0.5 \rfloor = -1.$$

Sketch the graph of $y = \lfloor x \rfloor$. Then compute:

1. $\lim\limits_{x \to 0^+} \lfloor x \rfloor$
2. $\lim\limits_{x \to 0^-} \lfloor x \rfloor$
3. $\lim\limits_{x \to 0} \lfloor x \rfloor$
4. $\lim\limits_{x \to 0} \lfloor x^2 \rfloor$

More limits from a graph



Find the value of

1. $\lim_{x \to 0^+} g(x)$
2. $\lim_{x \to 0^+} \lfloor g(x) \rfloor$
3. $\lim_{x \to 0^+} g(\lfloor x \rfloor)$
4. $\lim_{x \to 0^-} g(x)$
5. $\lim_{x \to 0^-} \lfloor g(x) \rfloor$
6. $\lim_{x \to 0^-} \lfloor \frac{g(x)}{2} \rfloor$
7. $\lim_{x \to 0^-} g(\lfloor x \rfloor)$

Exponential limits

Compute:

$$\lim_{t \to 0^+} e^{1/t}, \qquad \lim_{t \to 0^-} e^{1/t}.$$

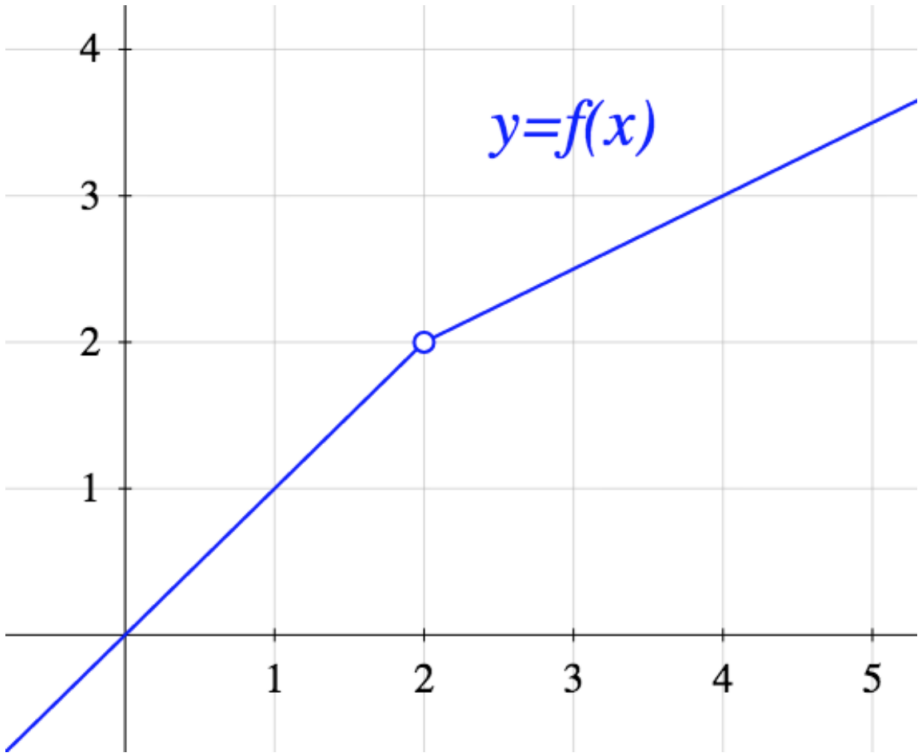
Suggestion: Sketch the graph of $y = e^x$ first.

Warm-up

Write down the formal definition of

$$\lim_{x \to a} f(x) = L.$$

Choosing δ



- Find one value of $\delta > 0$ s.t. $\quad 0 < |x-2| < \delta \Rightarrow |f(x)-2| < 0.5$
- Find **all** values of $\delta > 0$ s.t. $\quad 0 < |x-2| < \delta \Rightarrow |f(x)-2| < 0.5$

Infinite limits

Definition

Let $a \in \mathbb{R}$.

Let f be a function defined at least on an interval around a, except possibly at a.

Write a formal definition for

$$\lim_{x \to a} f(x) = \infty.$$

Implications

Let $a \in \mathbb{R}$. Let f be a function. Assume we know

$$0 < |x - a| < 0.1 \implies f(x) > 100$$

1. Which values of $\delta > 0$ satisfy .... ?

$$0 < |x - a| < \delta \implies f(x) > 100$$

2. Which values of $M \in \mathbb{R}$ satisfy ... ?

$$0 < |x - a| < 0.1 \implies f(x) > M$$

Preparation: choosing deltas

1. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

2. Find *all* values of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 1.$$

3. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < 0.1.$$

4. Let us fix $\varepsilon > 0$. Find a value of $\delta > 0$ such that

$$|x - 3| < \delta \implies |5x - 15| < \varepsilon.$$

Your first $\varepsilon - \delta$ proof

Goal

We want to prove that

$$\lim_{x \to 3} (5x + 1) = 16 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Write down a complete formal proof.

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement (3).
2. Write down what the structure of the formal proof should be, without filling the details.
3. **Rough work: What is δ?**
4. Write down a complete formal proof.

Is this proof correct?

Claim:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\quad 0 < |x| < \delta \implies |x^3 + x^2| < \varepsilon.$

Proof.

- Let $\varepsilon > 0$. Take $\delta = \sqrt{\dfrac{\varepsilon}{|x+1|}}$.
- Let $x \in \mathbb{R}$. Assume $0 < |x| < \delta$. Then

$$|x^3 + x^2| = x^2|x+1| < \delta^2|x+1| = \frac{\varepsilon}{|x+1|}|x+1| = \varepsilon.$$

- I have proven that $|x^3 + x^2| < \varepsilon$.

Choosing deltas again

We will go over this slide in the next class

Let us fix numbers $A, \epsilon > 0$.

1. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |Ax^2| < \varepsilon$
2. Find *many* values of $\delta > 0$ such that $|x| < \delta \Rightarrow |Ax^2| < \varepsilon$
3. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |x + 1| < 10$
4. Find *many* values of $\delta > 0$ such that $|x| < \delta \Rightarrow |x + 1| < 10$
5. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow \left\{ \begin{array}{c} |Ax^2| < \epsilon \\ |x + 1| < 10 \end{array} \right\}$
6. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |x^2 + x^3| < \varepsilon$

A harder proof

We will go over this slide in the next class

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{3}$$

directly from the definition.

1. Write down the formal definition of the statement (3).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. **Write down a complete formal proof.**