MAT137

(Section L0501, October 02, 2019)

- **For next day's lecture, watch videos 2.16 and 2.17**
- Today's lecture will **assume** you have watched videos 2.12, 2.13, 2.14 and 2.15.

Contents

1. Last day's limit example again!
2. Indeterminate form
3. A theorem about limits from definition
4. Squeeze theorem
5. New continuous functions
6. A new theorem about products and product rule

Choosing deltas again

Let us fix numbers $A, \epsilon > 0$.

1. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |Ax^2| < \varepsilon$
2. Find *many* values of $\delta > 0$ such that $|x| < \delta \Rightarrow |Ax^2| < \varepsilon$
3. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |x + 1| < 10$
4. Find *many* values of $\delta > 0$ such that $|x| < \delta \Rightarrow |x + 1| < 10$
5. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow \left\{ \begin{array}{c} |Ax^2| < \epsilon \\ |x+1| < 10 \end{array} \right\}$
6. Find a value of $\delta > 0$ such that $|x| < \delta \Rightarrow |x^2 + x^3| < \varepsilon$

A harder proof

Goal

We want to prove that

$$\lim_{x \to 0} \left(x^3 + x^2\right) = 0 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement (1).
2. Write down what the structure of the formal proof should be, without filling the details.
3. Rough work: What is δ?
4. **Write down a complete formal proof.**

Indeterminate form

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a.
Assume

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0.$$

What can we conclude about $\lim_{x \to a} \frac{f(x)}{g(x)}$?

1. The limit is 1.
2. The limit is 0.
3. The limit is ∞.
4. The limit does not exist.
5. We do not have enough information to decide.

Indeterminate examples

For each condition, give an example of a function f with domain $\mathbb{R}$ which satisfies that single condition.

1. $\lim_{x \to 0} \frac{f(x)}{x} = 0$
2. $\lim_{x \to 0} \frac{f(x)}{x} = 1$
3. $\lim_{x \to 0} \frac{f(x)}{x}$ does not exist and is not equal to $\pm\infty$
4. $\lim_{x \to 0} \frac{f(x)}{x} = \infty$

A theorem about limits

Theorem

Let f be a function with domain $\mathbb{R}$ such that

$$\lim_{x \to 0} f(x) = 3.$$

Prove that

$$\lim_{x \to 0} \left[5\, f(2x)\right] = 15$$

directly from the definition of the limit. Do not use any limit laws.

1. Write down the formal definition of what you want to prove.
2. Write down what the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
   Do you start from what you know and prove what you have to prove?
6. Did you remember not to assume your conclusion?

Squeeze to infinity

Squeeze Theorem

Let $a, L \in \mathbb{R}$.
Let f, g, and h be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x) \leq h(x)$
- $\lim_{x \to a} f(x) = L$
- $\lim_{x \to a} h(x) = L$

THEN $\lim_{x \to a} g(x) = L$.

Come up with a new version of this theorem which instead concludes that

$$\lim_{x \to a} g(x) = \infty$$

Hint: Draw a picture of the Squeeze Theorem. Then draw a picture of the new theorem.

Squeeze to infinity

Solve (4), (5) at home. If you have any difficulty talk to me

Squeeze Theorem (new version)

Let $a \in \mathbb{R}$.
Let f and g be functions defined near a, except possibly at a.
IF

- For x close to a but not a, $\quad f(x) \leq g(x)$
- $\lim_{x \to a} f(x) = \infty$

THEN $\lim_{x \to a} g(x) = \infty$.

1. Replace the first hypothesis with a more precise mathematical statement.
2. Write down the definition of what you want to prove.
3. Write down the structure of the formal proof.
4. Rough work
5. Write down a complete, formal proof.

Undefined function

Let $a \in \mathbb{R}$ and let f be a function. Assume $f(a)$ is undefined.

What can we conclude?

1. $\lim_{x \to a} f(x)$ exist
2. $\lim_{x \to a} f(x)$ doesn't exist.
3. No conclusion. $\lim_{x \to a} f(x)$ may or may not exist.

What else can we conclude?

4. f is continuous at a.
5. f is not continuous at a.
6. No conclusion. f may or may not be continuous at a.

A new function

We did not cover this in class. Go over this. If you have any difficulty, talk to me.

Let $a, b \in \mathbb{R}$. What does the following expression calculate?

$$f(a, b) = \frac{a + b + |a - b|}{2}$$

Prove it.

Hint: Plug in some values and guess.

Exercise: Find a similar expression for $\min\{a, b\}$.

New continuous functions

We did not cover this in class. Go over this. If you have any difficulty, talk to me.

We want to prove the following theorem

Theorem

IF f and g are continuous functions
THEN $h(x) = \max\{f(x), g(x)\}$ is also a continuous function.

You are allowed to use all results that we already know. What is the fastest way to prove this?

Hint: No epsilons are required.

True or false?

Is this claim true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$

A new theorem about products

Solve (4). We will go over this in the next class.

Theorem

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is **bounded**. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

Proof feedback

1. Is the structure of the proof correct?
   (First fix ε, then choose δ, then ...)
2. Did you say exactly what δ is?
3. Is the proof self-contained?
   (I do not need to read the rough work)
4. Are all variables defined? In the right order?
5. Do all steps follow logically from what comes before?
6. Did you only use things you know to be true?
   Did you prove what you have to prove?
7. Did you remember not to assume your conclusion?

Product rule

We did not cover this in class. Go over this. If you have any difficulty, talk to me.

Let $a \in \mathbb{R}$.

Let f and g be functions with domain $\mathbb{R}$, except possibly a.

Assume

- $\lim_{x \to a} f(x) = L$, and
- $\lim_{x \to a} g(x) = M$.

Show that g is bounded near a, that is

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

Hence show that $\lim_{x \to a} [f(x)g(x)] = LM$

(Hint: Write $f(x) = (f(x) - L) + L$ and use the previous theorem)