MAT137

(Section L0501, October 07, 2019)

- **For next day's lecture, watch videos (2.18,) 2.19, 2.20, 2.21 and 2.22**
- Today's lecture will **assume** you have watched videos 2.16, 2.17.

Contents

Another limit

Goal

We want to prove that

$$\lim_{x \to 1} x^3 = 1 \tag{1}$$

directly from the definition.

1. Write down the formal definition of the statement.
2. Write down what the structure of the formal proof should be, without filling the details.
3. **Rough work: What is** δ?
4. Write down a complete formal proof (Exc.).

How to write limits at infinity

Goal

We want to prove that

$$\lim_{x \to \infty} \frac{x+1}{x} = 1 \tag{2}$$

directly from the definition.

1. Write down the formal definition of the statement.
2. Write down what the structure of the formal proof should be, without filling the details.
3. **Rough work. What is M?**
4. Write down a complete formal proof (Exc.).

True or false?

Is this claim true?

Claim

Let $a \in \mathbb{R}$.

Let f and g be functions defined near a.

- IF $\lim_{x \to a} f(x) = 0$
- THEN $\lim_{x \to a} [f(x)g(x)] = 0$

A new theorem about products

Theorem

Let $a \in \mathbb{R}$.
Let f and g be functions with domain $\mathbb{R}$, except possibly a.
Assume

- $\lim_{x \to a} f(x) = 0$, and
- g is **bounded**. This means that

$$\exists M > 0 \text{ s.t. } \forall x \neq a, |g(x)| \leq M.$$

THEN $\lim_{x \to a} [f(x)g(x)] = 0$

1. Write down the formal definition of what you want to prove.
2. Write down the structure of the formal proof.
3. Rough work.
4. Write down a complete formal proof.

True or false?

Claim

Let $a, L \in \mathbb{R}$. IF f and g are functions such that

- (A) $\lim_{x \to a} f(x) = L$
- (B) $\lim_{y \to L} g(y) = M$

THEN

$$\lim_{x \to a} g(f(x)) = \lim_{y \to L} g(y) = M$$

This is FALSE.

Two ways to fix it (Do this as an exercise)

FALSE claim

Let $a, L \in \mathbb{R}$. IF f and g are functions such that

- (A) $\lim_{x \to a} f(x) = L$
- (B) $\lim_{y \to L} g(y) = M$

THEN

$$\lim_{x \to a} g(f(x)) = \lim_{y \to L} g(y) = M$$

Recall $\lim_{x \to a} f(x) = L$ means

$$\left\{ x \text{ close to } a \quad \text{and} \quad x \neq a \right\} \implies f(x) \text{ close to } L$$

And $\lim_{y \to L} g(y) = M$ means, replacing y by $f(x)$,

$$\left\{ f(x) \text{ close to } L \quad \text{and} \quad f(x) \neq L \right\} \implies g(f(x)) \text{ close to } M$$

1. Replace (B) with a condition to fix the theorem.
2. Alternatively, can you replace (A) with a condition to fix the theorem?

A difficult example

Construct a pair of functions f and g such that

- $\lim_{x \to 0} f(x) = 1$
- $\lim_{x \to 1} g(x) = 2$
- $\lim_{x \to 0} g(f(x)) = 42$

A question from an old test (Do this as an exercise)

The only thing we know about the function g is that

$$\lim_{x \to 0} \frac{g(x)}{x^2} = 2.$$

Use it to determine the following limits:

1. $\lim_{x \to 0} \frac{g(x)}{x}$
2. $\lim_{x \to 0} \frac{g(x)}{x^4}$
3. $\lim_{x \to 0} \frac{g(3x)}{x^2}$