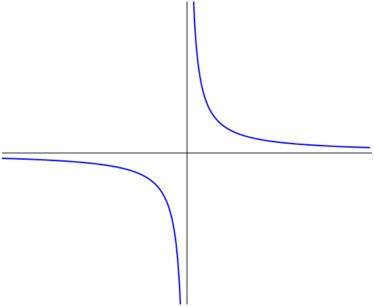
MAT137

(Section L0501, October 21, 2019)

- **For next day's lecture, watch videos 3.10, 3.11, 3.12.**
- Today's lecture will **assume** you have watched videos 3.6, 3.7, 3.9.

From the derivative to the function

1. Sketch the graph of a continuous function whose derivative has the graph below
2. Sketch the graph of a non-continuous function whose derivative has the graph below



More estimations

Without using a calculator, estimate $\sqrt[20]{1.01}$ as well as you can.

Hint: Consider the values you know for $f(x) = \sqrt[20]{x}$ and its derivative.

True or False

Let $a \in \mathbb{R}$.
Let f be a function with domain $\mathbb{R}$.
Assume f is differentiable everywhere.
What can we conclude?

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. f is continuous at a.
4. $f'(a)$ exists.
5. $\lim_{x \to a} f'(x)$ exists.
6. f' is continuous at a.

We will see a counterexample for (5) and (6) in next class!

Computing derivatives

Problem 1. Compute the derivatives of the following functions:

1. $f(x) = x^{100} + 3x^{30} - 2x^{15}$
2. $f(x) = \sqrt[3]{x} + 6$
3. $f(x) = \dfrac{4}{x^4}$
4. $f(x) = \sqrt{x}\,(1 + 2x)$
5. $f(x) = \dfrac{x^6 + 1}{x^3}$
6. $f(x) = \dfrac{x^2 - 2}{x^2 + 2}$

Problem 2. Let $0 \neq c \in \mathbb{R}$, and let f be a function that is differentiable at c. Define a new function g by:

$$g(x) = \frac{f(x)}{x^7}.$$

Compute $g'(c)$.

Proving the quotient rule.

Recall the quotient rule for derivatives from the videos, which I'll state formally here:

Theorem

Let $c \in \mathbb{R}$. Let f and g be functions defined at c and near c, and assume that $g(x) \neq 0$ for all x near c.
Define a function h by $h(x) = \dfrac{f(x)}{g(x)}$.

If f and g are differentiable at c, then h is differentiable at c, and

$$h'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$

First, use the definition of $h'(c)$ to write down the limit you need to prove.

Then prove it.

Recall a trick from the product rule.

In order to prove the product rule, we had to compute a similar limit, and to do that we did a simple "trick" of adding zero in a creative way:

$$\begin{aligned} &\frac{f(x)g(x) - f(c)g(c)}{x - c} \\ =&\frac{f(x)g(x) - f(c)g(x) + f(c)g(x) - f(c)g(c)}{x - c} \\ =&\frac{f(x) - f(c)}{x - c}\, g(x) + f(c)\frac{g(x) - g(c)}{x - c} \end{aligned}$$

A similar (but not identical) trick will help you with this proof.

Be careful to explicitly justify any limits you evaluate in your proof.

Check your proof of the quotient rule

1. Did you use the *definition* of the derivative?
2. Are there only equations and no words? If so, you haven't written a proof.
3. Does every step follow logically from the previous steps (with explanation)?
4. Did you assume anything you couldn't assume?
5. Did you assume at any point that a function is differentiatiable? If so, did you justify it?
6. Did you assume at any point that a function is continuous? If so, did you justify it?

If you answered "no" to Q6 above, your proof cannot be fully correct.

Critique this proof

$$
\begin{aligned}
h'(c) &= \lim_{x \to c} \frac{h(x) - h(c)}{x - c} = \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \\
&= \lim_{x \to c} \frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)\,(x - c)} \\
&= \lim_{x \to c} \frac{f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)}{g(x)g(c)\,(x - c)} \\
&= \lim_{x \to c} \left(\left[\frac{f(x) - f(c)}{x - c} g(c) - f(c) \frac{g(x) - g(c)}{x - c} \right] \frac{1}{g(x)g(c)} \right) \\
&= \left[f'(c)g(c) - f(c)g'(c) \right] \frac{1}{g(c)g(c)}
\end{aligned}
$$