MAT137

(Section L0501, February 12, 2020)

- **For today's lecture: slides 11.5-11.8**
- **For next day's lecture, watch videos 12.1–12.6 .**
- Contents: Sequences.

Recall: A sequence defined by recurrence

Consider the sequence R_n defined by

$$R_0 = 1$$
$$\forall n \in \mathbb{N}, \quad R_{n+1} = \frac{R_n + 2}{R_n + 3}$$

Compute R_1, R_2, R_3.

Is this proof correct?

Let R_n be the sequence in the previous slide.
Claim: $\{R_n\}_{n=0}^{\infty} \to -1 + \sqrt{3}$

Proof.

- Let $L = \lim_{n\to\infty} R_n$.
- $R_{n+1} = \frac{R_n + 2}{R_n + 3}$
- $\lim_{n\to\infty} R_{n+1} = \lim_{n\to\infty} \frac{R_n + 2}{R_n + 3}$
- $L = \frac{L+2}{L+3}$
- $L(L+3) = L + 2$
- $L^2 + 2L - 2 = 0$
- $L = -1 \pm \sqrt{3}$
- L must be positive, so $L = -1 + \sqrt{3}$

□

A corrected proof

Consider the sequence R_n defined by

$$R_0 = 1$$
$$\forall n \in \mathbb{N}, \quad R_{n+1} = \frac{R_n + 2}{R_n + 3}$$

1. Prove R_n is bounded below by 0.
2. Prove R_n is decreasing (use induction)
3. Prove R_n is convergent (use a theorem)
4. Now the calculation in the previous slide is correct, and we can get the value of the limit.

A famous sequence and da Vinci's painting

Consider the Fibonacci sequence (or Pingala sequence) F_n defined by

$$F_0 = 1, F_1 = 1$$
$$\forall n \geq 2, \quad F_n = F_{n-1} + F_{n-2}$$

1. What are the first few terms of the Fibonacci sequence?
2. Define for all $n \in \mathbb{N}$
$$R_n = \frac{F_{n+1}}{F_n}.$$
3. Assume $\lim_{n\to\infty} R_n = L$ exists and $L > 0$.
4. Find L.
5. L is a famous ratio in mathematics!!

Calculations

1. $$\lim_{n\to\infty} \frac{n! + 2e^n}{3n! + 4e^n}$$

2. $$\lim_{n\to\infty} \frac{2^n + (2n)^2}{2^{n+1} + n^2}$$

3. $$\lim_{n\to\infty} \frac{5n^5 + 5^n + 5n!}{n^n}$$

Big Theorem: TRUE or FALSE

Let a_n and b_n be positive sequences.

1. IF $a_n << b_n$ THEN $\forall m \in \mathbb{N},\ a_m < b_m$
2. IF $a_n << b_n$ THEN $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$
3. IF $a_n << b_n$
   THEN $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \Rightarrow a_m < b_m$

4. IF $\forall m \in \mathbb{N},\ a_m < b_m$ THEN $a_n << b_n$
5. IF $\exists m \in \mathbb{N}$ s.t. $a_m < b_m$ THEN $a_n << b_n$
6. IF $\exists n_0 \in \mathbb{N}$ s.t. $\forall m \in \mathbb{N},\ m \geq n_0 \Rightarrow a_m < b_m$ THEN $a_n << b_n$

Composition law

Write a proof for the following Theorem

Theorem

Let a_n be a sequence. Let $L \in \mathbb{R}$.

- IF $\begin{cases} a_n \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

1. Write the definition of your hypotheses and your conclusion.
2. Using the definition of your conclusion, figure out the structure of the proof.
3. Do some rough work if necessary.
4. Write a formal proof.

Composition law

Write a proof for the following Theorem

Theorem

Let a_n be a sequence. Let $L \in \mathbb{R}$.

- IF $\begin{cases} a_n \longrightarrow L \\ f \text{ is continuous at } L \end{cases}$
- THEN $\{f(a_n)\}_{n=0}^{\infty} \longrightarrow f(L)$.

Critique your proof!

1. Does your proof have the correct structure?
2. Did you introduce all your variables in the right order?
   Did you introduce all your variables?
3. Did you explain what you are doing?
4. Can the proof be fully understood without looking at your rough work?

An application of the previous theorem

Prove that

$$\sqrt{2+\sqrt{2+\sqrt{2+\dots}}}=2$$

1. Explicitly write the expression as the limit of a sequence $\{a_n\}_{n\in\mathbb{N}}$ (Hint: try to define by a recurrence)
2. Show that $0 \leq a_n \leq 2$ for all $n \in \mathbb{N}$.
3. Show that $\{a_n\}_{n\in\mathbb{N}}$ is increasing.
4. Conclude that $a := \lim_{n\to\infty} a_n$ exists.
5. Use that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$ and that $f(x) := \sqrt{2+x}$ is continuous at a to solve for a.