MAT137

(Section L0501, October 28, 2019)

- **For next day's lecture, watch videos 3.19, 3.20, 4.1, 4.2.**
- Today's lecture will **assume** you have watched videos 3.13, 3.15, 3.16, 3.17, 3.18 (and 3.14).
- Contents: 1) Implicit differentiation 2) exponentials and logarithms 3) Miscellaneous

Warm up 1: Rate of change

Let f be a continuous function with domain $\mathbb{R}$.

True or False?

IF the average rate of change of f between $x = 1$ and $x = 2$ is -3,
THEN f must be decreasing on $[1, 2]$.

True or False?

IF, for every $1 \leq x_1 < x_2 \leq 2$,
the average rate of change of f between x_1 and x_2 is negative,
THEN f must be decreasing on $[1, 2]$.

Warm up 2: Even more estimations

1. We know
$$f(0) = 2, \quad f'(0) = 3, \quad g(0) = 7, \quad g'(0) = 6.$$
Compute $\lim_{x \to 0} \frac{f(x)}{g(x)}$.

2. We know
$$f(0) = 0, \quad f'(0) = 3, \quad g(0) = 0, \quad g'(0) = 6.$$
   - When x is close to 0, give estimates for $f(x)$ and $g(x)$.
   - Use those estimates to compute $\lim_{x \to 0} \frac{f(x)}{g(x)}$.

Explicit functions

An equation like $y = x^2 + \sin(x)$ expresses a relationship between values of x and y.

More specifically, it says that the values of y that satisfy the equation are related to the values of x that satisfy the equation by a function.

(The function is $f(x) = x^2 + \sin(x)$.)

If we want to figure out how y varies when x varies, we can simply differentiate f, in this case getting

$$\frac{dy}{dx} = 2x + \cos(x).$$

Implicit functions

The equation $x^2 + y^2 = 1$ also expresses a relationship between values of x and y.

In this case though, the values of y cannot be expressed as an explicit function of x.

That is, there is no function f such that the equation $y = f(x)$ encapsulates all the information in the earlier equation.

But we still might want to ask how y varies when x varies.

Implicit functions

In the particular case of $x^2 + y^2 = 1$, we know that by splitting into two cases—when y is non-negative or non-positive—the relationship in each case can be expressed by an explicit function:

When $y \geq 0$, we know $y = \sqrt{1 - x^2}$.
When $y \leq 0$, we know $y = -\sqrt{1 - x^2}$.

We can also differentiate both of these functions to find out how y varies when x varies:

When $y \geq 0$, we find $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{y} \right)$.
When $y \leq 0$, we find $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{1 - x^2}} \left(= \dfrac{-x}{-\sqrt{1 - x^2}} = \dfrac{-x}{y} \right)$.

So, no matter what x is, it turns out that $\dfrac{dy}{dx} = \dfrac{-x}{y}$. This equation *accounts for both cases*.

Implicit differentiation

Instead of splitting up the cases, we could have done all of this at once by *implicitly differentiating* the original equation $x^2 + y^2 = 1$, as you saw in video 3.12.

To do this you differentiate both sides of the equation, and treat y as though it's a function of x.

So for example if you see a y^2, you apply the Chain Rule:

$$\frac{d}{dx}\left(y^2\right) = 2y\,y'.$$

In this case you'd get:

$$2x + 2y\,y' = 0 \implies y' = -\frac{x}{y}.$$

Notice that the RHS of this formula doesn't make sense when $y = 0$. That makes sense, since *y cannot be thought of as a function of x* around those points.

Can an implicit function be always written as a function?

Consider $x^2 + y^2 = 1$. For which values of (x, y) on the curve can y be expressed as a function of x in an interval around x?

For which values of (x, y) on the curve can x be expressed as a function of y in an interval around y?

Something strange happens at $(0, 0)$ for $x^2 - y^2 = 0$.

Some quick derivatives with exponentials and logarithms

Problem. Compute the derivatives of the following functions:

1. $f(x) = e^{\sin x + \cos x} \log(x)$
2. $f(x) = \pi^{\tan x}$
3. $f(x) = \ln\left[e^x + \ln(\ln(\ln(x)))\right]$

Reminder: We know:

- $\dfrac{d}{dx} e^x = e^x$
- $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$
- $\dfrac{d}{dx} a^x = a^x \ln a$