MAT137

(Section L0501, February 26, 2020)

- **For today's lecture: slides 12.7–12.10**
- **For next day's lecture, watch videos 13.1–13.4 .**
- Contents: BCT and LCT for improper integrals.

Quick review

For which values of $p \in \mathbb{R}$ is each of the following improper integrals convergent?

1. $\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$
2. $\displaystyle\int_0^1 \frac{1}{x^p}\, dx$
3. $\displaystyle\int_0^\infty \frac{1}{x^p}\, dx$

A "simple" integral

What is $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$?

1. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx = (\ln|x|)\Big|_{-1}^{1} = \ln|1| - \ln|-1| = 0$
2. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx = 0$ because $f(x) = \dfrac{1}{x}$ is an odd function.
3. $\displaystyle\int_{-1}^{1} \frac{1}{x}\, dx$ is divergent.

True or False - Part I

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that

$$\boxed{\forall x \geq a, \quad 0 \leq f(x) \leq g(x)}$$

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part II

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that

$$\boxed{\forall x \geq a, \quad f(x) \leq g(x)}$$

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

True or False - Part III

Let $a \in \mathbb{R}$.
Let f and g be continuous functions on $[a, \infty)$.
Assume that

$$\boxed{\exists M \geq a \text{ s.t. } \forall x \geq M, \quad 0 \leq f(x) \leq g(x)}$$

What can we conclude?

1. IF $\int_a^\infty f(x)dx$ is convergent, THEN $\int_a^\infty g(x)dx$ is convergent.
2. IF $\int_a^\infty f(x)dx = \infty$, THEN $\int_a^\infty g(x)dx = \infty$.
3. IF $\int_a^\infty g(x)dx$ is convergent, THEN $\int_a^\infty f(x)dx$ is convergent.
4. IF $\int_a^\infty g(x)dx = \infty$, THEN $\int_a^\infty f(x)dx = \infty$.

A simple BCT application

We want to determine whether $\displaystyle\int_1^\infty \frac{1}{x+e^x}\,dx$ is convergent or divergent.

We can try at least two comparisons:

1. Compare $\dfrac{1}{x}$ and $\dfrac{1}{x+e^x}$.
2. Compare $\dfrac{1}{e^x}$ and $\dfrac{1}{x+e^x}$.

Try both. What can you conclude from each one of them?

BCT calculations

Use the BCT to determine whether each of the following is convergent or divergent

1. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{2/3}}\,dx$
2. $\displaystyle\int_1^\infty \frac{1+\cos^2 x}{x^{4/3}}\,dx$
3. $\displaystyle\int_0^\infty \frac{\arctan x^2}{1+e^x}\,dx$
4. $\displaystyle\int_0^\infty e^{-x^2}\,dx$
5. $\displaystyle\int_2^\infty \frac{(\ln x)^{10}}{x^2}\,dx$

Rapid fire: convergent or ∞?

1. $\int_1^\infty \frac{1}{x^2}\,dx$
2. $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
3. $\int_1^\infty \frac{1}{x}\,dx$
4. $\int_1^\infty \frac{1}{x\sqrt{x}}\,dx$
5. $\int_0^1 \frac{1}{x^2}\,dx$
6. $\int_0^1 \frac{1}{\sqrt{x}}\,dx$
7. $\int_0^1 \frac{1}{x}\,dx$
8. $\int_1^\infty \frac{3}{x^2}\,dx$
9. $\int_1^\infty \frac{1}{x^2+3}\,dx$
10. $\int_1^\infty \left(\frac{1}{x^2}+3\right)dx$
11. $\int_1^\infty \left(\frac{1}{x^2+x}\right)dx$

A variation on LCT I

This is the theorem you have learned:

Theorem (Limit-Comparison Test)

Let $a \in \mathbb{R}$. Let f and g be positive, continuous functions on $[a, \infty)$.

- IF the limit $L = \lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ are both convergent or both divergent.

What if we change the hypotheses to $L = 0$ and $L = \infty$?

Hint: If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$, what is larger $f(x)$ or $g(x)$?
Prove it!

A variation on LCT II

Theorem (Limit-Comparison Test)

Let $a, b \in \mathbb{R}$. Let f and g be positive, continuous functions on $(a, b]$.

- IF the limit $L = \lim_{x \to a+} \frac{f(x)}{g(x)}$ exists and $L > 0$
- THEN $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ are both convergent or both divergent to ∞.

What should be the missing condition?

Prove as an exercise (Hint: same as the usual LCT proof)

Slow burn: convergent or divergent?

1. $\displaystyle\int_1^\infty \frac{x^3+2x+7}{x^5+11x^4+1}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2+x+1}}\,dx$
3. $\displaystyle\int_0^1 \frac{3\cos x}{x+\sqrt{x}}\,dx$
4. $\displaystyle\int_0^1 \cot x\,dx$
5. $\displaystyle\int_0^1 \frac{\sin x}{x^{3/2}}\,dx$
6. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx$