MAT137

(Section L0501, October 30, 2019)

- **For next day's lecture, watch videos 4.3, 4.4, 4.5.**
- Today's lecture will **assume** you have watched videos 3.19, 3.20, 4.1, 4.2.
- Contents: Logarithmic differentiation, related rates, inverse functions.

Some new limit computations!

Problem. Compute the following limits:

1. $\lim_{h \to 0} \frac{e^h - 1}{h}$.

2. $\lim_{x \to 0} \frac{e^{sin2x} - 1}{x}$.

True or false?

Problem. Let $f(x) = (x+1)^x$. Is the following formula true?

$$f'(x) = x \cdot (x+1)^{x-1}.$$

False! This formula is trying to use the power rule for a situation it can't be used for.

The power rule only applies to functions of the form $g(x) = x^{\text{constant}}$.

Logarithmic differentiation to the rescue! If we take log of both sides, we get:

$$\ln(f(x)) = \ln\left((x+1)^x\right) = x\,\ln(x+1),$$

which we can now differentiate implicitly and isolate for f'.

Logarithmic differentiation.

Problem. Compute the derivatives of the following functions:

1. $f(x) = (x+1)^x$.

2. $g(x) = x^{\tan(x)}$.

3. Now generalize these ideas into a new differentiation rule:

Let f and g be differentiable functions, and define h by

$$h(x) = [f(x)]^{g(x)}.$$

Derive a formula for $h'(x)$.

A very common error...

Problem. Calculate the derivative of

$$f(x) = (\sin x)^{\cos x} + (\cos x)^{\sin x}.$$

What is wrong with this answer?

$$\begin{aligned}
\ln f(x) &= (\cos x)\ln(\sin x) + (\sin x)(\ln\cos x) \\
\frac{d}{dx}\left[\ln f(x)\right] &= \frac{d}{dx}\left[(\cos x)\ln(\sin x)\right] + \frac{d}{dx}\left[(\sin x)(\ln\cos x)\right] \\
\frac{f'(x)}{f(x)} &= -(\sin x)\ln(\sin x) + (\cos x)\frac{\cos x}{\sin x} \\
&\quad + (\cos x)\ln(\cos x) + (\sin x)\frac{-\sin x}{\cos x}
\end{aligned}$$

$$f'(x) = f(x)\left[-(\sin x)\ln(\sin x) + (\cos x)\ln(\cos x) + \frac{\cos^2 x}{\sin x} - \frac{\sin^2 x}{\cos x}\right]$$

A different type of logarithm

Problem. Compute the derivative of

$$f(x) = \log_{x+1}\left(x^2 + 1\right).$$

Hint: If you don't know where to start, remember the definition of the logarithm:

$$\log_a b = c \iff a^c = b.$$

Congratulations!! You have now achieved full differentiation power!

With the tools you now know, you can more or less differentiate any function you can right down.

For example, you can compute the derivative of:

$$h(x) = \sqrt[3]{\frac{(\sin^6 x)\sqrt{x^7 + 6x + 2}}{3^x (x^{10} + 2x)^{10}}}$$

It will be long, but easy. Taking a log of both sides will turn the right side into a long sum, which is easy to differentiate.

Lake ripple: Do this as an exercise

We drop a pebble into a lake. It produces a circular ripple. When the radius is 2 meters and is increasing at a rate of 10 cm/s, at what rate is the area increasing?

Sliding ladder

A ten-meter long ladder is leaning against a vertical wall and sliding. The top end of the ladder is 8 meters high and sliding down at a rate of 1 meter per second.

At what rate is the bottom end sliding?

Math party

The MAT137 TAs wanted to rent a disco ball for their upcoming party. However, they could only afford a flashlight. At the party, one TA is designated the "human disco ball". The TA stands in the center of the room pointing the flashlight horizontally and spins at 3 revolutions per second. The room is square with side length 8 meters.

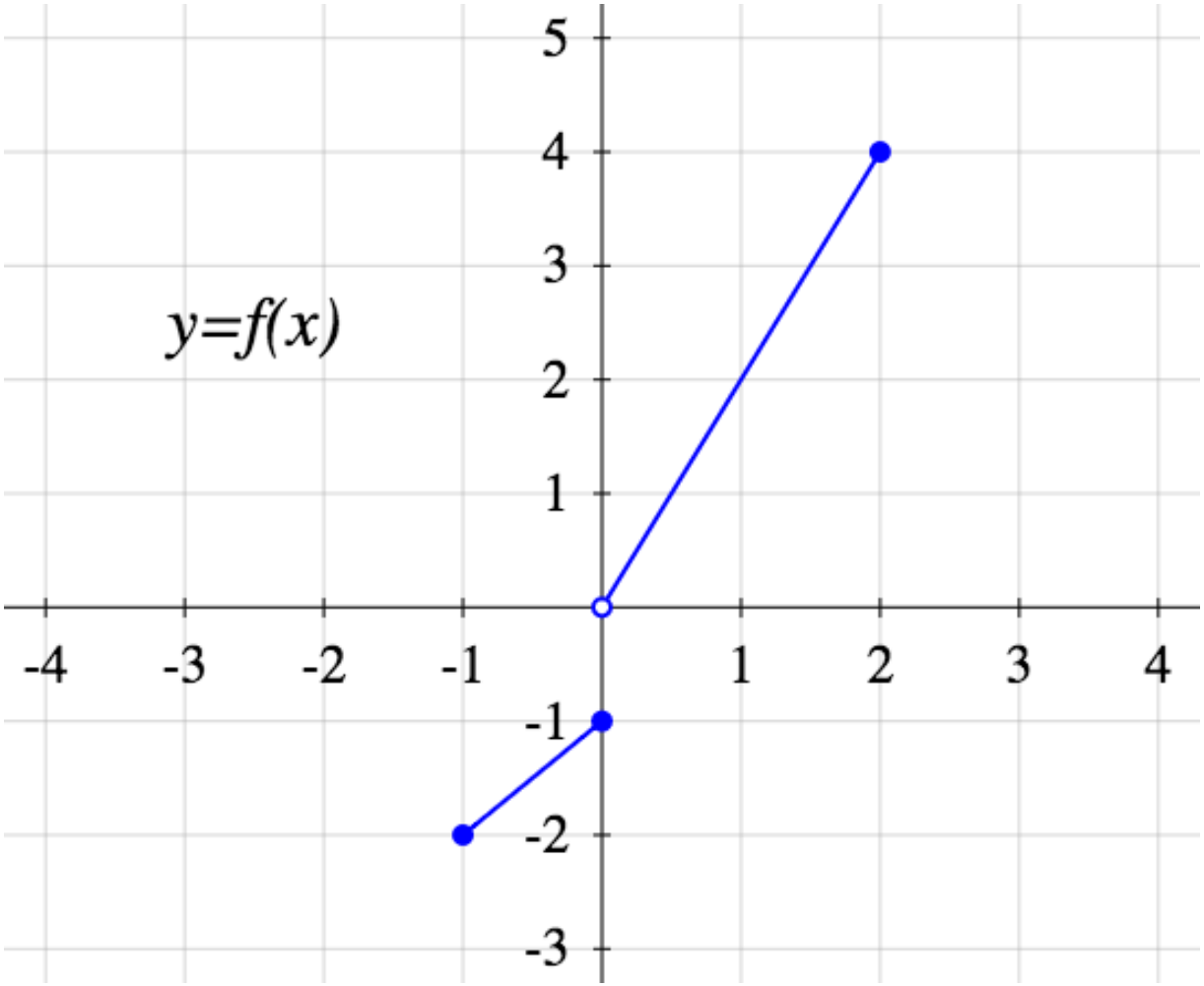
At what speed is the light from the flashlight moving across the wall when it is 2 meters away from a corner?

Sleepy ants: Do this as an exercise

Two ants are taking a nap. The first one is resting at the tip of the minute hand of a cuckoo clock, which is 25 cm long. The second one is resting at the tip of the hour hand, which is half the length.

At what rate is the distance between the two ants changing at 3:30?

Inverse function from a graph



Calculate:

1. $f(2)$
2. $f(0)$
3. $f^{-1}(2)$
4. $f^{-1}(0)$
5. $f^{-1}(-1)$

Absolute value and inverses

Let

$$h(x) = x|x| + 1$$

1. Calculate $h^{-1}(-8)$.
2. Find an equation for $h^{-1}(x)$.
3. Sketch the graphs of h and h^{-1}.
4. Verify that for every $x \in \boxed{???}$, $h(h^{-1}(x)) = x$
5. Verify that for every $x \in \boxed{???}$, $h^{-1}(h(x)) = x$.