MAT137

(Section L0501, March 04, 2020)

- **For today's lecture: slides 13.5–13.9**
- **For next day's lecture, watch videos 13.10–13.12 .**
- Contents: Properties of series.

More rapid fire review

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$

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2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$
4. $\displaystyle\int_1^\infty \frac{x+1}{x^3+2}\,dx$
5. $\displaystyle\int_1^\infty \frac{\sqrt{x^2+5}}{x^3+6}\,dx$
6. $\displaystyle\int_1^\infty \frac{x^2+3}{\sqrt{x^5+2}}\,dx$

Geometric series

Calculate the value of the following series:

1. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots$
2. $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \dots$
3. $\frac{3}{2} - \frac{9}{4} + \frac{27}{8} - \frac{81}{16} + \dots$
4. $1 + \frac{1}{2^{0.5}} + \frac{1}{2} + \frac{1}{2^{1.5}} + \frac{1}{2^2} + \frac{1}{2^{2.5}} + \dots$
5. $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{2^{2n+1}}$
6. $\sum_{n=k}^{\infty} x^n$

Proving a property about series from the definition

Let $\sum_{n=0}^{\infty} a_n$ be a series. Let $c \in \mathbb{R}$. Prove that

- IF $\sum_{n=0}^{\infty} a_n$ is convergent.
- THEN $\sum_{n=0}^{\infty} (ca_n)$ is also convergent and

$$\sum_{n=0}^{\infty} (ca_n) = c \left[\sum_{n=0}^{\infty} a_n \right].$$

Write a proof directly from the definition of series.

Is $0.999999\cdots = 1$?

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1. Write the number $0.9999999\ldots$ as a series
   Hint: $427 = 400 + 20 + 7$.

2. Compute the first few partial sums

3. Add up the series.
   Hint: it is geometric.

Series expansion

From the geometric series, we know that when $|x| < 1$, we can expand the function $f(x) = \dfrac{1}{1-x}$ as a series:

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Find a similar way to write the following functions as series:

$$g(x) = \frac{1}{1+x^2} \qquad h(x) = \frac{1}{2-x}$$

Rapid fire: convergent or divergent?

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$

Rapid fire: convergent or divergent?

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3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^{n/2}}$
4. $\displaystyle\sum_{n=5}^{\infty} \frac{3^n}{2^{2n+1}}$
5. $\displaystyle\sum_{n=3}^{\infty} \frac{3^n}{1000 \cdot 2^{n+2}}$
6. $\displaystyle\sum_{n=0}^{\infty} (-1)^n$

Necessary condition: TRUE or FALSE

1. IF $\lim_{n\to\infty} a_n = 0$, THEN $\sum_{n}^{\infty} a_n$ is convergent.
2. IF $\lim_{n\to\infty} a_n \neq 0$, THEN $\sum_{n}^{\infty} a_n$ is divergent.
3. IF $\sum_{n}^{\infty} a_n$ is convergent THEN $\lim_{n\to\infty} a_n = 0$.
4. IF $\sum_{n}^{\infty} a_n$ is divergent THEN $\lim_{n\to\infty} a_n \neq 0$.

Examples

A series $\sum_{n=0}^{\infty} a_n$ may be any one of:

- convergent
- divergent to ∞
- divergent to $-\infty$
- divergent and "oscillating"

1. Give one example of each of the four results.

Examples

A series $\sum_{n=0}^{\infty} a_n$ may be any one of:

- convergent
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1. Give one example of each of the four results.
2. Now assume $\forall n \in \mathbb{N}, a_n \geq 0$.
   Which of the four outcomes is still possible?

What can you conclude?

Assume $\forall n \in \mathbb{N}, a_n > 0$. Consider the series $\sum_{n=0}^{\infty} a_n$.
Let $\{S_n\}_{n=0}^{\infty}$ be its sequence of partial sums.

In each case, what can you conclude about the **series**? Is it convergent, divergent, or we do not know?

1. $\forall n \in \mathbb{N}, \exists M \in \mathbb{R}$ s.t. $S_n \leq M$.
2. $\exists M \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}, S_n \leq M$.
3. $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, a_n \leq M$.
4. $\exists M > 0$ s.t. $\forall n \in \mathbb{N}, a_n \geq M$.

Series expansion again

Using geometric series, we showed last time that

$$g(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \qquad \text{for } |x| < 1$$

1. Compute $\frac{d}{dx} \arctan x$
2. Pretend you can take antiderivatives of series just like you do with finite sums. Which series adds up to $\arctan x$?

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1. Compute $\frac{d}{dx} \arctan x$
2. Pretend you can take antiderivatives of series just like you do with finite sums. Which series adds up to $\arctan x$?
3. Find the value of

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}$$