MAT137

(Section L0501, November 20, 2019)

- **For next day's lecture, watch videos 6.1, 6.2.**
- Today's lecture will **assume** you have watched videos 5.7-5.12.
- Contents: MVT.

Recap: A nice consequence of Rolle's Theorem

Theorem

Let $a < b$ be real numbers. Let f be a function defined on $[a, b]$.

IF

- *f is continuous on $[a, b]$ and differentiable on (a, b)*
- *f is **not** injective on $[a, b]$*

THEN $\exists c \in (a, b)$ *such that* $f'(c) = 0$.

1. Write the definition of "f is not injective on $[a, b]$". You will need it.
2. Recall the statement of Rolle's Theorem. You will need that too.
3. Do some rough work to understand why this is true.
4. Write the proof.

Warm up

- Let f be a function defined on an interval I. Write the definition of "f is increasing on I".

- Write the statement of the Mean Value Theorem.

Positive derivative implies increasing in an open interval

Use the MVT to prove

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

1. Recall the definition of what you are trying to prove.
2. From that definition, figure out the structure of the proof.
3. If you have used a theorem, did you verify the hypotheses?
4. Are there words in your proof, or just equations?

Is this proof correct?

Theorem

Let $a < b$. Let f be a differentiable function on (a, b).

- IF $\forall x \in (a, b), f'(x) > 0$,
- THEN f is increasing on (a, b).

Proof.

- From the MVT, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- We know $b - a > 0$ and $f'(c) > 0$
- Therefore $f(b) - f(a) > 0$, so $\quad f(b) > f(a)$
- f is increasing.

$\square$

Positive derivative implies increasing on a closed interval

What conditions would you need here?

Theorem

Let $a < b$ and f be differentiable on ..., continuous on ...,

- IF $\forall x \in$ ..., $f'(x) > 0$,
- THEN f is increasing on $[a, b]$.

Let $a \in \mathbb{R}$. What conditions on f will guarantee that

1. f is increasing on (a, ∞)?
2. f is increasing on $[a, \infty)$?
3. f is increasing on $(-\infty, a)$?
4. f is increasing on $(-\infty, a]$?

Proof is same as before.

Let $g(x) = x^3(x^2-4)^{1/3}$.

Find out on which intervals this function is increasing or decreasing.
Using that information, sketch its graph.

To save time, here is the first derivative:

$$g'(x) = \frac{x^2(11x^2-36)}{3(x^2-4)^{2/3}}$$

Cauchy's MVT - Part 1

Here is a new theorem:

We want to prove this Theorem

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF (some conditions)
THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

What is wrong with this "proof"?

- By MVT, $\exists c \in (a, b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- By MVT, $\exists c \in (a, b)$ s.t. $g'(c) = \dfrac{g(b) - g(a)}{b - a}$
- Divide the two equations and we get what we wanted.

$\square$

Cauchy's MVT - Part 2

We want to prove this theorem

Let $a < b$. Let f and g be functions defined on $[a, b]$.
IF

- f and g are continuous on $[a, b]$,
- f and g are differentiable on (a, b),
- $g(b) \neq g(a)$

THEN $\exists c \in (a, b)$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

1. There is one number $M \in \mathbb{R}$ so that you will be able to apply Rolle's Theorem to the new function $H(x) = f(x) - Mg(x)$ on the interval $[a, b]$. What is M?
2. Apply Rolle's Theorem to H. What do you conclude?
3. Fill in the missing hypotheses in the theorem above.
4. Prove it.

Prove that, for every $x \in \mathbb{R}$

$$e^x \geq 1 + x$$

Hint: When is the function $f(x) = e^x - 1 - x$ increasing or decreasing?