MAT137

(Section L0501, January 13, 2020)

- **Fot today's lecture: slides 7.5-7.9**
- **For next day's lecture, watch videos 7.10, 7.11, 7.12, 8.1, 8.2 .**
- Contents: integrability.

Example 1: non-continuous

Let $f(x) = \begin{cases} 0 & x = 0 \\ 5 & 0 < x \leq 1 \end{cases}$, defined on $[0, 1]$.

1. Let $P = \{0, 0.2, 0.5, 0.9, 1\}$.
   Calculate $L_P(f)$ and $U_P(f)$ for this partition.
2. Let $P = \{x_0, x_1, \ldots, x_N\}$ be any partition of $[0, 1]$.
   What is $U_P(f)$? What is $L_P(f)$? (Draw a picture!)
3. Find a partition P such that $L_P(f) = 4.99$.
4. What is the upper integral, $\overline{I_0^1}(f)$?
5. What is the lower integral, $\underline{I_0^1}(f)$?
6. Is f integrable on $[0, 1]$?

A very tricky question from last class

Let f be a bounded function on $[a, b]$.
Which statement is true?

1. There exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_P(f).$$

2. There exist partitions P and Q of $[a, b]$ such that

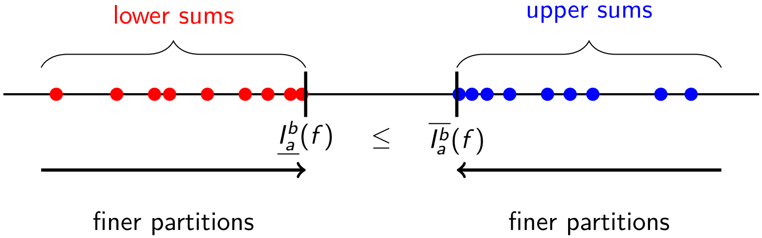
$$\underline{I_a^b}(f) = L_P(f) \qquad \text{and} \qquad \overline{I_a^b}(f) = U_Q(f).$$

True or False

Let f be a bounded function on $[a, b]$.

3. There exists a partition P of $[a, b]$ such that

$$\underline{I_a^b}(f) = L_P(f)$$

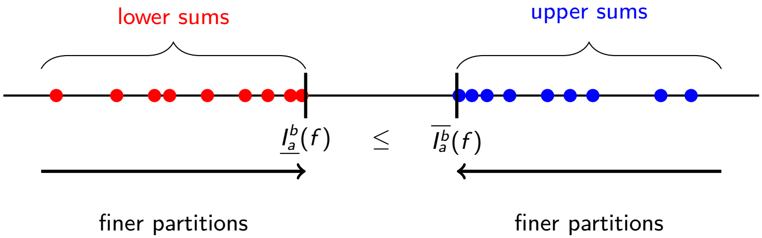


The "ε-characterization" of integrability

True or False?

Let f be a bounded function on $[a, b]$.

1. IF f is integrable on $[a, b]$
   THEN $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.
2. IF $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$,
   THEN f is integrable on $[a, b]$



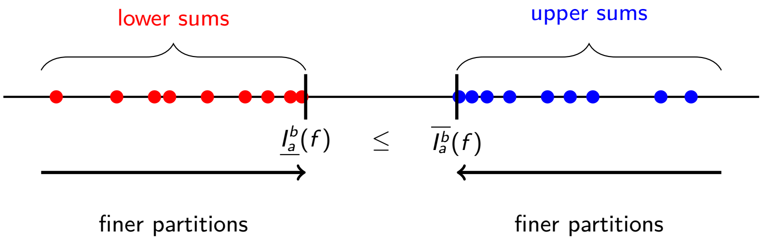
"ε-characterization" of integrability

Let's prove this theorem from last slide.

Theorem

Let f be a bounded function on $[a, b]$.

1. IF $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$,
   THEN f is integrable on $[a, b]$
2. IF f is integrable on $[a, b]$
   THEN $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$.



"ε-characterization" $\Rightarrow$ integrable

Claim 1

Let f be a bounded function on $[a, b]$.
IF $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$
THEN f is integrable on $[a, b]$

Prove this claim. *Hints:*

1. Recall the definition of "f is integrable on $[a, b]$".
2. Let P be a partition.
   Order the quantities $U_P(f)$, $L_P(f)$, $\overline{I_a^b}(f)$, $\underline{I_a^b}(f)$.
   (Draw a picture on the real line)
3. Order $U_P(f) - L_P(f)$, $\overline{I_a^b}(f) - \underline{I_a^b}(f)$, and 0.

integrable $\Rightarrow$ "ε-characterization"

Claim 2

Let f be a bounded function on $[a, b]$.
IF f is integrable on $[a, b]$
THEN $\forall \varepsilon > 0$, $\exists$ a partition P of $[a, b]$, s.t. $U_P(f) - L_P(f) < \varepsilon$

Prove this claim. *Hints:*

1. Assume f is integrable on $[a, b]$.
   Let I be the integral. Fix $\varepsilon > 0$.
2. Recall the definition of "f is integrable on $[a, b]$".
3. There exists a partition P_1 s.t. $U_{P_1}(f) < I + \frac{\epsilon}{2}$. Why?
4. There exists a partition P_2 s.t. $L_{P_2}(f) > I - \frac{\epsilon}{2}$. Why?
5. What can you say about $U_{P_1}(f) - L_{P_2}(f)$?
6. Can you construct a partition P such that
   $L_{P_2}(f) \leq L_P(f) \leq U_P(f) \leq U_{P_1}(f)$?