If it looks and smells like the reals...

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Abstract

Given a topological space $< X,\ \mathcal{T} > \in M$, an elementary submodel of set theory, we define X_M to be $X \cap M$ with topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. We prove that if X_M is homeomorphic to $\mathbb{R}$, then $X = X_M$. The same holds for arbitrary locally compact uncountable separable metric spaces, but is independent of ZFC if "local compactness" is omitted.

Given a model of set theory, i.e. a collection W of sets which satisfies the usual set-theoretic axioms (ZFC), a set $M \subseteq W$ is an *elementary submodel* of W if for every natural number n and for every formula φ with n free variables in the predicate calculus with $=$ and a 2-place relation symbol $\in$, and every $x_1, \cdots, x_n \in M$ (we will systematically confuse the membership relation and the symbol '$\in$'), $\varphi(x_1, \cdots, x_n)$ holds in M if and only if it does in W. We usually think of W as being V, the universe of all sets, but for technical reasons officially deal with $W = H(\theta)$, the collection of all sets of hereditary cardinality less than θ, a "sufficiently large" regular uncountable cardinal and rather than dealing with ZFC, we deal with sufficiently large fragments of it. (For more on these technical reasons, see [JW].) The non-logician reader will not lose much by thinking of elementary submodels of V.

Elementary submodels have been used in set-theoretic topology with increasing frequency and depth over the past 20 years (see e.g. [D]). As often happens in mathematics, one's tools become objects of study; thus in [JT] we inaugurated a systematic investigation of the topological spaces induced by elementary submodels. This paper is a continuation of that study, although it is mainly independent of [JT].

The *Downward Löwenheim-Skolem Theorem* of Logic implies that, given any set $X \in H(\theta)$ and an infinite cardinal $\kappa \leq |H(\theta)|$, there is an elementary submodel M of $H(\theta)$ with $X \in M$ and $|M| = \kappa$. Given a topological space $<X, \mathcal{T}> \in M$, we define X_M to be the space $X \cap M$ with topology $\mathcal{T}_M$ generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. The Downward Löwenheim-Skolem Theorem yields X_M's with $X \cap M$ having any infinite cardinality $\leq |X|$; a natural question is whether an *Upward Löwenheim-Skolem Theorem* holds in this context, i.e.

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given a space $<X, \mathcal{T}>$, is it equal or perhaps homeomorphic to Y_M's for Y's of arbitrary cardinality $> |X|$, and suitable M's. We shall show that this in general false, but that it is true in some special cases. Along the way, we come across some perhaps unexpected rigidity properties of familiar spaces, e.g.

Theorem 1. *If X_M is homeomorphic to $\mathbb{R}$, so is X.*

Subsets of $\mathbb{R}$ are sufficient to illustrate diversity with respect to such rigidity:

Theorem 2.

a) *For every infinite cardinal κ, there is an X of size κ and an M such that X_M is homeomorphic to $\mathbb{Q}$.*

b) *It is independent of ZFC (modulo large cardinals) whether there is an X such that X_M is homeomorphic to a subspace of $\mathbb{R}$ of size $\aleph_1$ but X_M is not homeomorphic to X (or even to a subspace of $\mathbb{R}$).*

Except for some excursions into large cardinals, our proofs will use little more than the definition of elementary submodels, plus classic topology that can be found in [E]. (We will refer to [E] rather than refer to the original authors and papers.) Thus this paper is intended to be accessible both to logicians and to topologists. Before proving a generalization of Theorem 1 we give a particularly elementary proof of the next result, which illustrates our methods.

Theorem 3. *If X_M is an uncountable compact metric space, then $X_M = X$.*

Theorem 3 will be derived as a corollary to

Theorem 4. *If $[0, 1] \subseteq M$ and if X_M is a hereditarily separable, hereditarily Lindelöf T_3 space, then $X_M = X$.*

To prove Theorem 4, we need several lemmas.

Lemma 5.

a) *X_M Hausdorff implies X Hausdorff.*

b) *X_M regular implies X regular.*

Proof. The first is left to the reader. For the second it suffices, by elementarity, to show $M \models X$ is regular, i.e. that

$$(\forall x \in X \cap M)(\forall U \in \mathcal{T} \cap M)(x \in U \cap M \to (\exists V \in \mathcal{T} \cap M)$$
$$(x \in V \cap M \ \& \ \forall y \in X \cap M)[(\forall W \in \mathcal{T} \cap M)(y \in W \to W \cap V \neq \emptyset) \to y \in U]))$$

But since the topology on X_M is generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$, this is equivalent to saying X_M is regular.

Lemma 6. *Suppose $X, Y \in M$, $|X| \geq |Y|$ and $M \supseteq X$. Then $M \supseteq Y$.*

Proof. There is an injection $g : Y \to X$. Hence there is an injection $g \in M$ such that $g : Y \to X$. Suppose $y \in Y$ and $x = g(y)$. Then $x \in M$ so y is definable in M as the unique z such that $<z, x> \in g$, so $y \in M$.

Recall

Definition. $\{x_\alpha\}_{\alpha<\kappa} \subseteq X$ is *left-separated (right-separated)* if there exist open $\{U_\alpha\}_{\alpha<\kappa}$ such that for every α, $x_\alpha \in U_\alpha$, but for all $\beta > \alpha$ $(\beta < \alpha)$ $x_\alpha \notin U_\beta$.

Lemma 7. *(see e.g. [R]). A space X is hereditarily separable (hereditarily Lindelöf) if and only if it includes no uncountable left (right)-separated subspace.*

Lemma 8. *If $|\omega_1 \cap M| = \aleph_1$ and X_M is hereditarily separable (hereditarily Lindelöf), so is X.*

Proof. Suppose X is not hereditarily separable. Then there is an injection $f : \omega_1 \to X$ such that range f is left-separated. By elementarity, there is such an $f \in M$, and if $|\omega_1 \cap M| = \aleph_1$, this gives us a left-separated subspace of size $\aleph_1$ in X_M. Similarly for hereditarily Lindelöf.

Actually, "$|\omega_1 \cap M| = \aleph_1$" is equivalent to "$\omega_1 \subseteq M$", but we don't need this here.

If X is hereditarily separable (hereditarily Lindelöf), so is X_M but we don't need this here either. [JT] is concerned with going from properties of X to those of X_M; here we do the converse. Of course the difference is purely conceptual.

Proof of Theorem 4. By Lemma 6, $\omega_1 \subseteq M$ so X is hereditarily Lindelöf and hereditarily separable. Since X is Hausdorff by Lemma 5a) and hereditarily Lindelöf, $|X| \leq 2^{\aleph_0}$, so since X is (hereditarily) separable and by 5b) regular, X has a basis of size $\leq 2^{\aleph_0}$. By hereditary Lindelöfness again, $|\mathcal{T}| \leq 2^{\aleph_0}$, so by Lemma 6, X and $\mathcal{T}$ are included in M, so $X_M = X$.

Now we move on to the proof of Theorem 3.

Proof. We use 5 classical results, two from [JT], and a new one.

Lemma 9. *[E, 1.7.11]. Every uncountable compact metric space includes a closed dense-in-itself subspace.*

Lemma 10. *[E, 4.5.5(a)]. Every compact metric dense-in-itself space includes a copy of the Cantor set $\mathbb{K}$.*

Lemma 11. *[E, 4.5.9(b)]. There is a continuous surjection from $\mathbb{K}$ to [0,1].*

Lemma 12. *[E, 2.1.8]. Any continuous function from a closed subspace of a normal space into $[0,1]$ can be extended over the whole space.*

Definition. A continuous function is *perfect* if it sends closed sets to closed sets and if each point-inverse is compact.

Lemma 13. *[E, 3.7.2]. The preimage of a compact space under a perfect map is compact.*

Lemma 14. *[JT]. If X is locally compact T_2, X_M is the image of a subspace of X under a perfect map.*

Lemma 15. *[JT]. For a first countable space X, X_M coincides with the subspace topology on $X \cap M$.*

The final lemma is due to Lucia R. Junqueira and is included with her kind permission.

Lemma 16. *If X_M is compact, so is X.*

PROOF. Suppose X has an open cover $\mathcal{U}$ that has no finite subcover. Then, by elementarity, there is a $\mathcal{U} \in M$ such that M thinks $\mathcal{U}$ is an open cover of X with no finite subcover. Then $\{U \cap M : U \in \mathcal{U} \cap M\}$ is an open cover of X_M and hence has a finite subcover $\{U \cap M : \ U \in \mathcal{U}'\}$, where $\mathcal{U}'$ is a finite subset of $\mathcal{U}$. Then $\mathcal{U}' \in M$ and M thinks $\mathcal{U}'$ covers X, so it does.

Putting these together, let $Z \subseteq X$ and $\pi : Z \to X_M$ be perfect and onto. Let $L \subseteq X_M$ be homeomorphic to $\mathbb{K}$. Let $f : L \to [0,1]$ be onto. Then, since X is normal, $f \circ (\pi|\pi^{-1}L) : \pi^{-1}(L) \to [0,1]$ extends to a g mapping X onto $[0,1]$. By elementarity, there is an onto $g \in M,\ g : X_M \to [0,1]_M$. But by Lemma 15, $[0,1]_M = [0,1] \cap M$ with the subspace topology, so $[0,1] \cap M$ is compact. It includes $\mathbb{Q}$, so it $= [0,1]$. Thus $[0,1] \subseteq M$.

REMARK. We should mention that although X_M is not in general compact even if X is, there are examples of X's such that X_M is compact and yet $X_M \neq X$. For example, let X be a one-point compactification of a discrete space and let $|M| < |X|$.

We next improve Theorem 3 to get

Theorem 17. *If X_M is a locally compact hereditarily Lindelöf uncountable Hausdorff space, then $X_M = X$.*

Theorem 1 is then an immediate corollary. Indeed any separable metric space is hereditarily Lindelöf.

PROOF. We first need to show

Lemma 18. *If X_M is locally compact, so is X.*

PROOF. By elementarity, noting that finite subsets of members of M are in M, it suffices to show

$$
\begin{aligned}
&(\forall x \in X \cap M)(\forall U \in \mathcal{T} \cap M)(\exists V \in \mathcal{T} \cap M)[x \in V \cap M \& \\
&(\forall y \in X \cap M)([(\forall W \in \mathcal{T} \cap M)(y \in W \cap M \to W \cap V \cap M \neq 0)] \to y \in U) \& \\
&(\forall \mathcal{S} \in M)(\mathcal{S} \subseteq \mathcal{T} \cap M \ \& \ (\forall y \in X \cap M)[(\forall W \in \mathcal{T} \cap M)(y \in W \to \\
&W \cap V \cap M \neq \emptyset) \to (\exists S \in \mathcal{S})(y \in S)] \to (\exists \text{ finite } \mathcal{S}' \subseteq \mathcal{S}) \\
&(\forall y \in X \cap M)[(\forall W \in \mathcal{T} \cap M)(y \in W \to W \cap V \cap M \neq \emptyset] \to \\
&(\exists S \in \mathcal{S}')(y \in S)]
\end{aligned}
$$

But since X_M is locally compact, we have this.

Proof of Theorem 17. Since X_M is locally compact and hereditarily Lindelöf, it is σ-compact and first countable. Since it is σ-compact and uncountable, it includes an uncountable compact first countable subspace. By a standard Cantor-Bendixson argument, X_M then has an uncountable compact first countable subspace without isolated points. Such a subspace maps onto $[0,1]$ [J, proof of 3.16]. As in the proof of Theorem 3, we then get a compact subspace L of X and a map f from L onto $[0,1]$. We then apply elementarity to

Lemma 19. [E, 3.1.C]. *If L is compact Hausdorff and $f : L \to Y$ is a continuous (and hence closed if Y is T_2) surjection, then there is a closed $L' \subseteq L$ such that $f|L'$ maps L' onto Y but no proper closed subset of L' is mapped by f onto Y.*

to get an $F \in M$ such that $F \cap M$ is closed in X_M, there is a continuous surjection g from $F \cap M$ (as a subspace of X_M) to $[0,1] \cap M$ such that if $H \in M$ is a closed subset of X, then $g(F \cap H \cap M)$ is closed in $[0,1] \cap M$, and if $g(F \cap H \cap M) = [0,1] \cap M$, then $F \cap H \cap M = F \cap M$.

$F \cap M$ is a closed subspace of a locally compact Hausdorff space and so is locally compact and hence satisfies the Baire Category Theorem. We claim $[0,1] \cap M$ does also. It suffices to show that if V is dense open in $[0,1] \cap M$, then $g^{-1}(V)$ is dense open in $F \cap M$. If so, given $\{V_n\}_{n<\omega}$ dense open in $[0,1] \cap M$, take $x \in \bigcap_{n<\omega} g^{-1}(V_n)$. Then $g(x) \in \bigcap_{n<\omega} V_n$. To show $g^{-1}(V)$ is dense open, take $W \in \mathcal{T} \cap M$ such that $W \cap F \cap M \neq \emptyset$. If $F \cap M \subseteq W \cap M$, then $g^{-1}(V) \cap W \cap M \neq \emptyset$, so suppose $F \cap M - W \cap M \neq \emptyset$. Then $g((F - W) \cap M) \neq [0,1] \cap M$, so there is a $y \in V \cap [0,1] \cap M - g((F - W) \cap M)$. Take $x \in F \cap M$ such that $g(x) = y$. Then $x \in g^{-1}(V) \cap W \cap M$.

$F \cap M$ is a closed subspace of a σ-compact space so it is σ-compact, say $F \cap M = \bigcup_{n<\omega} F_n$, F_n compact. Then for some n, $g(F_n)$ is a compact somewhere dense subset of $[0,1] \cap M$. Therefore there are $q < r \in \mathbb{Q} \cap [0,1]$ such that $(q,r) \cap M \subseteq g(F_n)$. But $(q,r) \cap M$ is dense in (q,r), so $g(F_n) \supseteq [q,r]$. But then $M \supseteq [0,1]$ and we can finish off as in the proof of Theorem 3. We need only recall that the weight (least cardinal of a base) of a locally compact Hausdorff space does not exceed its cardinality [E, 3.3.6] so "local compactness" can substitute for "hereditary separability" in Theorem 4.

Remark. We have in effect proved that closed irreducible images of Baire spaces are Baire, as was noted in [AL]. I thank E. Michael for supplying the reference.

After seeing this proof, S. Todorcevic came up with a considerably shorter and simpler one which just uses the proof for the compact uncountable metric case, but I decided the technique of the proof given here is sufficiently interesting to justify its inclusion.

Uncountability is necessary in Theorem 3, since if we take a countable M, $(\omega_1 + 1)_M$ is a compact metric space. Also observe

Theorem 20. *For any infinite regular X without isolated points, there is an M such that X_M is homeomorphic to $\mathbb{Q}$.*

PROOF. Take a countable elementary submodel M containing X. Then X_M is regular [JT], has no isolated points, is countable, and has a countable base. But countable metric spaces without isolated points are homeomorphic to $\mathbb{Q}$ [E, 6.2.A.(d)].

Hereditary Lindelöfness – or some countability condition – is necessary in Theorem 17, else we could take the discrete space of size $\aleph_2$ and then take an elementary submodel of size $\aleph_1$. An example which is better – since X has no isolated points and $M \supseteq [0,1]$ – is to take the disjoint sum of $(2^{\aleph_0})^+$ copies of $[0,1]$ and then take a countably closed elementary submodel of size $2^{\aleph_0}$. Then X_M is the sum of $2^{\aleph_0}$ copies of $[0,1]$, so is a locally compact uncountable metric space, but is not equal to X.

For general uncountable separable metric spaces, we enter the realm of large cardinals . For example,

Theorem 21.

a) *If $2^{\aleph_0} = \aleph_1$ and $0^\sharp$ does not exist, then if X_M is an uncountable separable metric space, $X_M = X$.*

b) *If Chang's Conjecture holds, there is a non-metrizable X such that X_M is an uncountable separable metric space.*

$0^\sharp$ is a set of natural numbers, the existence of which has large cardinal strength. The non-existence of $0^\sharp$ is equivalent to *Jensen's Covering Lemma for L*, which is more familiar to set-theoretic topologists. $V = L$ implies $0^\sharp$ does not exist. See [K] for details. Theorem 21a) follows quickly from Lemma 6, Theorem 4, and

Lemma 22 [KT]. *If $0^\sharp$ does not exist and $|M| \geq \kappa$, then $\kappa \subseteq M$.*

Proof of Theorem 21a). Since X_M and hence M is uncountable, $\omega_1 \subseteq M$. By CH and Lemma 6, $[0,1] \subseteq M$. By Theorem 4, we are done.

Definition. *Chang's Conjecture* (see e.g. [K]) is the assertion that every model M of size $\aleph_2$ with a distinguished subset S of size $\aleph_1$ has an elementary submodel N of size $\aleph_1$ such that $|N \cap S| = \aleph_0$.

Proof of Theorem 21b). The proof divides into two cases, depending on the size of $2^{\aleph_0}$. First, assume $2^{\aleph_0} \geq \aleph_2$ and Chang's Conjecture. Take an elementary submodel M of size $\aleph_2$ of some sufficiently large $H(\theta)$, with $\mathbb{R} \in M$ and $\mathbb{R} \subseteq M$. Expand $<M, \in>$ to the model $<M, \in, \omega_1>$ which distinguishes ω_1. Take an elementary submodel $<N, \in, \omega_1>$ of $<M, \in, \omega_1>$ with $|N| \geq \aleph_1$ and $|N \cap \omega_1| = \aleph_0$. Now $N \models |\mathbb{R}| > \omega_1$, so $|N \cap \mathbb{R}| = \aleph_1$. Thus if L is the long line, L_N is separable, uncountable, and metrizable although L is not.

Chang's Conjecture plus $2^{\aleph_0} = \aleph_2$ follows from *Martin's Maximum* [FMS], which is consistent if there is a supercompact cardinal. Later, L.R. Junqueira came up with another example using the same hypothesis, which has the advantage of being compact, although it is not first countable as is the long line. It is simply the product of $\aleph_1$ copies of the two-point discrete space. When I presented my example in Toronto, S. Todorcevic informed me that , using a result of Tarski [T], Baumgartner [B] had constructed in ZFC a linear order of density $\aleph_1$ and size $\aleph_1^\mu$ where μ is the least cardinal such that $\aleph_1^\mu > \aleph_1$. The order is obtained in the usual way from the branches of length μ of a certain tree. The corresponding linearly ordered topological space X has character μ; thus if CH holds, the space has character $\aleph_1$ and so is not metrizable. On the other hand, its Chang Conjecture reflection will be an uncountable separable linearly ordered space X_M. This does not quite assure metrizability, but we can modify X by sticking in a copy of $\mathbb{Q}$ between any two adjacent points. This changes none of the relevant cardinal functions, but now the resulting X_M will have a countable base. As a bonus, it turns out that separable linearly ordered metrizable spaces are embeddable in $\mathbb{R}$ [E, 6.3.2(c)] so whether or not CH holds, we obtain

Corollary 23. *Chang's Conjecture implies there is a non-metrizable X such that X_M is homeomorphic to a subspace of $\mathbb{R}$.*

The point is that the L_N above has cardinality $\aleph_1 < 2^{\aleph_0}$ and is therefore 0-dimensional and so embeds in the Cantor set.

Remark. The long line provides an interesting counterexample to the topological metatheorem which asserts that "homeomorphic" is the same as "equal" as far as topology is concerned. We have seen that the long line L can have an L_N which is homeomorphic to a subspace of $\mathbb{R}$, although L is not. On the other hand, suppose we have a space $<X, \mathcal{T}>$ such that for some M, X_M is actually a subspace of $\mathbb{R}$. Since $\mathbb{R}$ and its topology are definable, $M \models <X, \mathcal{T}>$ is a subspace of $\mathbb{R}$, so it is.

The conclusion of Theorem 21a) does not follow from the non-existence of $0^\sharp$.

Theorem 24. *It's consistent that $2^{\aleph_0} = \aleph_2$, $0^\sharp$ does not exist, and there is an M such that $\mathbb{R}_M$ is not homeomorphic to $\mathbb{R}$.*

Proof. Simply add say $\aleph_2$ Cohen reals to a model of $V = L$. Then $0^\sharp$ does not exist because it cannot be added by set forcing (see e.g. [K]). Then in the extension simply take M to be any elementary submodel of size $\aleph_1$ including $\aleph_1$ reals of some sufficiently large $H(\theta)$, with $\mathbb{R} \in M$.

Under CH, I. Farah proved all uncountable $\mathbb{R}_M$'s are equal to $\mathbb{R}$. See [KT].

If $|X_M| = 2^{\aleph_0}$, we do not need CH in Theorem 21a) so we have e.g.

Corollary 25. *If $0^\sharp$ does not exist and X_M is homeomorphic to an uncountable Borel subspace of $\mathbb{R}$, then $X_M = X$.*

I do not know if the non-existence of $0^\sharp$ is necessary, even for $\mathbb{R} - Q$. Assuming $0^\sharp$ does not exist, if $|X_M| = \aleph_1$ and X_M is separable metric, then as in the proof of Theorem 4, we have all finite powers of X are hereditarily Lindelöf and hereditarily separable, so X has a $G_\delta-$diagonal. Gary Gruenhage has shown (private communication) that nonetheless X need not be metrizable.

Theorem 21a) cannot be improved to drop separability:

Example. It is consistent with CH and $0^\sharp$ doesn't exist that there is a non-metrizable first countable space of size $\aleph_2$ such that X_M is metrizable for every M of size $\aleph_1$.

Proof. $V = L$ implies there is a stationary $E \subseteq \{\alpha \in \omega_2 : cf(a) = \omega\}$ such that $E \cap \alpha$ is not stationary in α, for any $\alpha \in \omega_2$.

It follows that every subspace of E of cardinality $< \aleph_2$ is metrizable, but E is not.

Constructing a ladder system on E (see e.g. [F]), one obtains a space with the additional properties that it is a locally compact, locally separable Moore space.

On the other hand,

Theorem 26. *If for every M such that $|X \cap M| \leq \aleph_1$, X_M is separable metrizable, then X is separable metrizable.*

Proof. This is essentially proved in 3.2 of [D]. Actually, we only require one special M:

Definition. M is *$\omega-$covering* if every countable subset of M is included in a member of M.

One can construct an $\omega-$covering elementary submodel of $H(\theta)$ with $<X, \mathcal{T}> \in M$ as $\bigcup\limits_{\alpha<\omega_1} M_\alpha$, where $<X, \mathcal{T}> \in M_0$, a countable elementary submodel of $H(\theta)$, $M_{\alpha+1} \supseteq M_\alpha \cup \{M_\alpha\} \cup \{x_\alpha\}$, $M_{\alpha+1}$ a countable elementary submodel of $H(\theta)$, $\{x_\alpha : \alpha < \omega_1\} \subseteq X$, and α limit implies $M_\alpha = \bigcup\limits_{\beta<\alpha} M_\beta$. Since X_M has a countable base, there is a countable subset $\mathcal{B}$ of $\{U \cap M : U \in \mathcal{T} \cap M\}$ which is a base, since $\{U \cap M : U \in \mathcal{T} \cap M\}$ is a base. But by $\omega-$covering, we may assume $\mathcal{B} \in M$. Then $M \models X$ has a countable base, so it does.

Corollary 27. $2^{\aleph_0} = \aleph_1$ *if and only if whenever $<X, \mathcal{T}>$ is a space such that for every M such that $|X \cap M| \leq \aleph_1, X_M$ is homeomorphic to a subspace of $\mathbb{R}$, so is X.*

Proof. Assuming CH, since by Theorem 26, X is separable metric and therefore $|X \cup \mathcal{T}| \leq \aleph_1$ we may take an $M \supseteq X \cup \mathcal{T}$, $|M| = \aleph_1$. Then $X = X_M$ so X is homeomorphic to a subspace of $\mathbb{R}$.

On the other hand, suppose $2^{\aleph_0} > \aleph_1$. Take an elementary submodel M of $H(\theta)$ of size $\aleph_1$ containing $\mathbb{R}$. Then $(\mathbb{R} \times \mathbb{R})_M$ is a separable 0-dimensional metric space and hence embeddable in $\mathbb{R}$, yet $\mathbb{R} \times \mathbb{R}$ is not embeddable in $\mathbb{R}$.

An E as above shows that Theorem 26 is consistently not true if one drops "separability". It will be difficult to construct just in ZFC a non-metrizable X such that all X_M with $|X \cap M| \leq \aleph_1$ are metrizable. The reason is that L.R.

Junqueira has shown (unpublished) that if there is an ω-covering M such that X_M is metrizable, then X is first countable and all subsets of size $\leq \aleph_1$ are metrizable. No example in ZFC is known of such a non-metrizable X.

Assuming the existence of a supercompact cardinal, there *is* a topological Upward Löwenheim-Skolem Theorem for large spaces:

Theorem 28. *Suppose $|X| \geq \kappa$, a supercompact cardinal, say $<X, \mathcal{T}> \in H(\theta)$. Then for every $\lambda \geq \theta + |X \cup \mathcal{T}|$ there is a $<Y, \mathcal{S}>$, $|Y| \geq \lambda$, a Ψ and an elementary submodel M of $H(\Psi)$ such that $<Y, \mathcal{S}> \in M$ and $<X, \mathcal{T}>$ is homeomorphic to Y_M.*

PROOF. Take a supercompact embedding j with $j(\kappa) \geq \lambda$. Take $M = j''H(\theta)$. Then M is an elementary submodel of $H(j(\theta))$. Let $Y = j(X)$ and $\mathcal{S} = j(\mathcal{T})$. Then $<X, \mathcal{T}>$ is homeomorphic to Y_M, which is just $j''X$ with topology $\{j''U : U \in \mathcal{T}\}$.

One can attempt to carry out this construction in case we have a sufficiently closed elementary embedding j existing in some generic extension of V. In this case, however, $Y_M \notin V$ and is no longer homeomorphic to $<X, \mathcal{T}>$ $-$ which is no longer a topological space $-$ but rather to the topology on X generated by $\mathcal{T}$ in that generic extension.

Let me end by restating the most interesting remaining open problem.

PROBLEM Is it a theorem of ZFC that if X_M is homeomorphic to $\mathbb{R} - \mathbb{Q}$, then $X = X_M$?

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