CHARACTERIZING ω_1 AND THE LONG LINE BY THEIR TOPOLOGICAL ELEMENTARY REFLECTIONS

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ABSTRACT. Given a topological space $\langle X, \mathcal{T} \rangle \in M$, an elementary submodel of set theory, we define X_M to be $X \cap M$ with the topology generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. We prove that it is undecidable whether X_M homeomorphic to ω_1 implies $X = X_M$, yet it is true in ZFC that if X_M is homeomorphic to the long line, then $X = X_M$. The former result generalizes to other cardinals of uncountable cofinality while the latter generalizes to connected, locally compact, locally hereditarily Lindelöf T_2 spaces.

0. INTRODUCTION

We take M to be an elementary submodel of H_θ for θ a sufficiently large regular cardinal, but act as if $H_\theta = V$. For an extended discussion of this standard circumlocution see [3] or [5] or [8].

Let $\langle X, \mathcal{T} \rangle$ be a topological space which is a member of M. Let X_M be $X \cap M$ with topology $\mathcal{T}_M$ generated by $\{U \cap M : U \in \mathcal{T} \cap M\}$. In the abstract, in [8] the second author proved that if X_M is homeomorphic to $\mathbb{R}$, then $X = X_M$. K. Kunen asked if analogous results hold for ordinals. The first section of this paper, which forms part of the University of Toronto Ph.D thesis of the first author [6], written under the supervision of the second author, shows that this is true for cardinals under an additional hypothesis, but is undecidable in general, even for ω_1. This renders the second section result - due to the second author - quite surprising, namely that if X_M is homeomorphic to the long line, i.e. $\omega_1 \times \mathbb{R}$ ordered lexicographically and given the order topology, then $X = X_M$.

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We need the following result:

Theorem 0.1. [2] *Let $\langle X, \mathcal{T}\rangle$ be a locally compact T_2 space and let M be a elementary submodel such that $\langle X, \mathcal{T}\rangle \in M$. Then there is a $Y \subseteq X$ and $\pi : \langle Y, \mathcal{T}\rangle \longrightarrow X_M$ such that π is perfect and onto.*

The mapping is defined as follows:

Let

$\mathcal{V}_x = \{V \in \mathcal{T} \cap M : x \in V\}$, for $x \in X_M$.

$K_x = \bigcap \mathcal{V}_x$, for $x \in X_M$.

Note that, since X is Hausdorff, a simple elementary submodel argument shows that if $x, y \in M$ and $x \neq y$, then $K_x \cap K_y = \emptyset$.

Define

$$Y = \bigcup \{K_x : x \in X_M\},$$

and

$$\pi : \langle Y, \mathcal{T}\rangle \longrightarrow \langle X_M, \mathcal{T}_M\rangle,$$

by

$$\pi(y) = x \text{ if and only if } y \in K_x.$$

□

1. Upwards reflection of cardinal spaces

We first solve the easier question of what happens when X_M is actually equal to an ordinal.

Theorem 1.1. *Let κ be an ordinal, $\langle X, \mathcal{T}\rangle$ a topological space and let M be an elementary submodel such that $X, \mathcal{T}, \kappa \in M$. If $X_M = \kappa$ then $X = \kappa$.*

Proof:

Notice that as $X_M = \kappa$:

(1) $M \models (\forall x, y \in X)\ (x \in y \text{ or } y \in x)$.
(2) $M \models (\forall x \in X)$ (x is an ordinal).
(3) $M \models (\forall x \in X)\ (\forall y \in x)\ (y \in X)$.
(4) $M \models (\forall A \subseteq X)$ (A has an $\in -$minimal element).

All of the above imply that $M \models X$ is an ordinal, which implies that $H(\theta) \models X$ is an ordinal. Notice that this fact holds in the set sense, i.e. we still have to prove that it is also true in the topological sense. Also notice that we used here that $\in$ is an absolute relation, so the same arguments would not hold for "homeomorphic" instead of "equal".

Claim 1: For $\gamma_1 < \gamma_2 \in \kappa$ we have that $M \models (\gamma_1, \gamma_2]$ is open in X.

Proof: Otherwise we would have

$$M \models (\exists \gamma_1, \gamma_2 < \kappa)\ (\exists x \in I = (\gamma_1, \gamma_2] \setminus int((\gamma_1, \gamma_2])).(*)$$

Pick $\gamma_1, \gamma_2, x \in M$ satisfying the above sentence. We may say that $x = \gamma < \kappa$. As $I \in M$, $x \in I$ and $X_M = \kappa$, I is open in X_M. So there is a $V \in \mathcal{T}_M$ such that $x \in V \cap M \subseteq I$. Therefore $M \models V$ is open, $x \in V$ and $V \subseteq I$. This contradicts $(*)$ and we have the result.

$\square$

Claim 2: $M \models (\forall V \in \mathcal{T})\ (\forall \alpha \in X)\ (\alpha \in V \to (\exists \beta < \alpha)\ ((\beta, \alpha] \subseteq V))$.

Proof: Pick $V \in \mathcal{T} \cap M$. $V \cap M \in \mathcal{T}_M$. Using that $X_M = \kappa$ topologically we may find $\beta < \alpha$ such that $(\beta, \alpha] \subseteq V \cap M$. As before, $(\beta, \alpha] \in M$ and so we have that $M \models (\beta, \alpha] \subseteq V$.

$\square$

Claim 1 and Claim 2 combined say that $M \models X$ is homeomorphic to an ordinal, and consequently $H(\theta) \models X$ is homeomorphic to an ordinal. All we need to prove now is that this ordinal is actually κ.

Suppose not. Then $X = \lambda > \kappa$. Since $\kappa \subseteq M$ and $\kappa \in M$, we would have $X \cap M = \lambda \cap M$, which includes $\kappa + 1$. This would contradict $X \cap M = \kappa$.

$\square$

We will now state and prove a version of the Theorem when we have "homeomorphic" instead of "equal".The "equals" version would follow from a positive "homeomorphic" version, but for the latter, we only have consistency results and even that not for all ordinals. As we do not have the topology on the space defined by an absolute order, $\in$, the proof technique is rather different.

First we need the following Lemmas; we give a hint of proof for those not yet published.

Lemma 1.2. *Let κ be a cardinal of cofinality $\geq \omega_1$. Consider the topology induced by the order and let $Y \subseteq \kappa$ be such that with the subspace topology, Y is homeomorphic to κ. Then Y is a closed subset of κ.*

Proof: Fix $h : Y \longrightarrow \kappa$ a homeomorphism and notice that since they are homeomorphic,

$$|Y| = \kappa\ (*).$$

Suppose that Y is not closed. Pick $\alpha = min\{\theta : \theta \in cl(Y) \setminus Y\}$.

Claim :

$$h''(Y \cap [0, \alpha]) \text{ is bounded in } \kappa\ (**).$$

To see that, suppose otherwise. Then $C_1 = h''(Y \cap [0, \alpha])$ is a closed unbounded subset of κ. In this case $C_2 = h''(Y \setminus [0, \alpha])$ is a closed subset of κ which must be bounded as $cof(\kappa) \geq \omega_1$, and for such κ two closed unbounded sets must meet, but C_1 and C_2 are obviously disjoint.

So for some β, $C_2 \subseteq [0, \beta]$, which is compact, and consequently $Y \setminus [0, \alpha] \subseteq h^{-1}([0, \beta])$ which is also compact. So $Y \setminus [0, \alpha] \subseteq [0, \gamma] \subseteq \kappa$, for some $\gamma < \kappa$.

Notice now that this implies $Y = (Y \cap [0, \alpha]) \cup (Y \setminus [0, \alpha]) \subseteq [0, \max(\alpha, \gamma)]$, which implies $|Y| < \kappa$, contradicting $(*)$.
□

Now, let $\lambda = cof(\alpha)$. Fix $f : \lambda \longrightarrow \alpha$ a cofinal mapping.

By induction, we will construct a strictly increasing sequence $\sigma = \{\alpha_\epsilon : \epsilon < \lambda\} \subseteq Y$ converging to $\alpha \notin Y$ with the following induction hypothesis:

(1) $\forall \beta < \lambda$, $\{\alpha_\epsilon : \epsilon < \beta\}$ is strictly increasing.
(2) If $\beta < \lambda$ is a limit ordinal, then $\alpha_\beta = sup\{\alpha_\epsilon : \epsilon < \beta\}$.

Choose $\alpha_0 \in (f(0), \alpha] \cap Y$. If α_ϵ is chosen, then:

$$\alpha_{\epsilon+1} \text{ is chosen in } (\alpha_\epsilon, \alpha] \cap (f(\epsilon + 1), \alpha] \cap Y \ (***).$$

It remains to choose α_β for $\beta < \lambda$ a limit ordinal. Notice that $\sigma_\beta = \{\alpha_\epsilon : \epsilon < \beta\}$ is not closed in κ. If it were closed, as it is bounded, it would have an upper bound in σ_β, and this would contradict the fact that it is strictly increasing. It has then a limit point and this limit point has to be in Y by the minimality of α and since $\beta < cof(\alpha)$. We choose α_β to be this limit point. By $(***)$, σ converges to α.

Notice that by the way we constructed it, $\sigma \cup \{\alpha\}$ is closed in κ. So σ is closed in Y.

We have that $h''(\sigma)$ is closed in κ and is bounded in κ by (**). It is then a compact subset of κ. As h is a homeomorphism, σ is compact in Y. As Y is a subspace of κ, it is compact in κ. This is a contradiction since it includes a sequence converging to a point outside of it.
□

Lemma 1.3. *(see [1]) Suppose that M is an elementary submodel and X is a topological space such that $\langle X, \mathcal{T} \rangle \in M$. If X_M is compact then X is compact and X_M is a perfect image of X.*

This and the following lemma are not hard to prove, using elementarity and the perfect map defined after the statement of Theorem 0.1.

Lemma 1.4. *(see [8]) Suppose that M is an elementary submodel and X is a topological space such that $\langle X, \mathcal{T}\rangle \in M$. If X_M is locally compact T_2 then X is also.*

Lemma 1.5. *(see [5]) Suppose that M is an elementary submodel, $X, Y \in M$ and $Y \subseteq M$. If $|X| \leq |Y|$ then $X \subseteq M$.*

This is straightforward, taking the function witnessing $|X| \leq |Y|$ to be in M.

Let $\chi(x, X)$ be the least cardinal of a neighborhood base at x.

Lemma 1.6. [2] *If M is an elementary submodel, $\kappa \subseteq M$, and $\langle X, \mathcal{T}\rangle \in M$ is a topological space such that for every $x \in X$, $\chi(x, X) \leq \kappa$, then X_M is a subspace of X.*

Theorem 1.7. *Let M be an elementary submodel, $\langle X, \mathcal{T}\rangle$ a topological space and κ a cardinal with $cof(\kappa) \geq \omega_1$ and such that $\kappa, X, \mathcal{T} \in M$ and also $\kappa \subseteq M$. If X_M is homeomorphic to κ, then $X = X_M$ and hence is homeomorphic to κ.*

Proof: As X_M is homeomorphic to κ, it is locally compact T_2. So by Lemma 1.4, X is locally compact T_2 and we can use Theorem 0.1 to get a surjective perfect map $\pi : Z \subseteq X \longrightarrow X_M$. Fix $f : \kappa \longrightarrow X_M$ a homeomorphism and call $x_\alpha = f(\alpha)$.

Then:

$$(\forall \alpha < \kappa)\ (\exists V_\alpha \in \mathcal{T} \cap M)\ (x_\alpha \in V_\alpha \cap M \subseteq f''([x_0, x_\alpha])).$$

Notice that:

$$|V_\alpha \cap M| < \kappa\ (*).$$

Claim 1: $V \models |V_\alpha| < \kappa$.

Otherwise $M \models |V_\alpha| \geq \kappa$ which implies $M \models (\exists B \subseteq V_\alpha)(|B| = \kappa)$. Pick such a $B \in M$. Since $\kappa \subseteq M$, by Lemma 1.5 we get that $B \subseteq M$. So $B \subseteq V_\alpha \cap M$ has size κ, which contradicts $(*)$.

□

Claim 2 : $V \models (\forall x \in X)\ (\exists V \in \mathcal{T})\ (x \in V$ and $|V| < \kappa)$.

It is enough to prove that M models the previous sentence.

Suppose otherwise, that:

$$M \models (\exists x \in X)\ (\forall V \in \mathcal{T})\ ((x \in V) \rightarrow (|V| \geq \kappa))\ (**).$$

Pick such an $x \in X \cap M$. There is some α such that $x = x_\alpha$ and so the previously defined V_α contradicts $(**)$.

□

Now, $V \models$ Every point in X has a neighborhood of size $< \kappa$. Since X is locally compact T_2, we get that each point in X has a neighborhood base of size $< \kappa$. Since $\kappa \subseteq M$, it follows that $X_M = X \cap M$ has the subspace topology inherited from X (by Lemma 1.6).

Claim 3: $X \cap M$ is open in X. To see that, for $x \in X \cap M$, use Claim 2 and get V a neighborhood of x of size $\lambda < \kappa$. This V may be taken in M (as $x \in M$) and so by Lemma 1.5 we have that $V \subseteq M$. Now, $x \in V \subseteq X \cap M$.

□

We will prove now that $|X| \leq \kappa$. Suppose otherwise that:

$$V \models |X \setminus X \cap M| \geq \kappa \ (***).$$

We obtain: $V \models (\exists \text{ open } Y \subseteq X)(Y \cong \kappa \text{ and } |X \setminus Y| \geq \kappa)$. So,

$$V \models (\exists \text{ open } Y \subseteq X)(Y \cong \kappa \text{ and } (\exists T \subseteq X \setminus Y)(|T| = \kappa).$$

By elementarity, $M \models (\exists \text{ open } Y \subseteq X)(Y \cong \kappa$ and

$(\exists T \subseteq X \setminus Y)(|T| = \kappa)$.

Pick $Y, T \in M$.

Again, by Lemma 1.5, $T \subseteq X \cap M$ and $Y \subseteq X \cap M$. In particular Y is open in $X \cap M$.

Remember that $X \cap M = X_M$ which is homeomorphic to κ. So fix $g : \kappa \longrightarrow X \cap M$ a homeomorphism.

$C = g^{-1}(Y)$ is a subset of κ homeomorphic to κ. So it is closed in κ by Lemma 1.2. As Y is open, C is also open in κ. So $D = \kappa \setminus C$ is a closed subset of κ. Because two clubs in κ always meet (as $cof(\kappa) \geq \omega_1$), D is a bounded set in κ having then cardinality less than κ. The contradiction comes from the fact that $g^{-1}(T) \subseteq D$ and has size κ. So $(***)$ is false which implies $|X| = \kappa$, and by Lemma 1.5 $X \subseteq M$, so $X = X \cap M$ and hence is homeomorphic to κ.

□

We do not know if our results can be extended to singular cardinals of countable cofinality, or to arbitrary ordinals of uncountable cofinality.

The assumption that $\kappa \subseteq M$ is a strong one; however the following two theorems will show that the condition that the cardinal be included in the model cannot be eliminated from the hypothesis of Theorem 1.7, at least when κ is not weakly inaccessible.

Theorem 1.8. *Suppose that M is an elementary submodel, κ is a cardinal and $\langle X, \mathcal{T}\rangle$ is a topological space such that $X, \mathcal{T}, \kappa, \kappa^+ \in M$ and $\kappa \subseteq M$ but $\kappa^+ \not\subseteq M$. If X_M is homeomorphic to κ^+ then X is not homeomorphic to κ^+.*

Proof:

We have two cases. If for some $x \in X$ $\chi(x, X) \geq \kappa^+$, then trivially X is not homeomorphic to κ^+. Suppose then that for every $x \in X$, $\chi(x, X) \leq \kappa$. Then by Lemma 1.6, X_M is a subspace of X.

If X is homeomorphic to κ^+ , fix $f : \kappa^+ \longrightarrow X = \{x_\alpha : \alpha < \kappa^+\}$ a homeomorphism. We may pick $f \in M$. Notice that $Y = f^{-1}(X \cap M)$ is a subset of κ^+ homeomorphic to κ^+. By Lemma 1.2, Y is closed in κ^+, which implies $X \cap M$ is closed in X.

As $\kappa^+ \not\subseteq M$, X is not a subset of M (again just use that $f \in M$). Pick now $\gamma = min\{\alpha : x_\alpha \notin X \cap M\}$. Since $f \in M$, we have that γ is an infinite limit ordinal. Now $\sigma = \{x_\alpha : \alpha < \gamma\} \subseteq X \cap M$ and $x_\gamma \in cl(\sigma)$ as $\gamma \in cl(\{\alpha : \alpha < \gamma\}$ and f is a homeomorphism. Since $X \cap M$ is closed in X, $x_\gamma \in X \cap M$, which contradicts the definition of γ.

$\square$

Theorem 1.9. *Suppose that M is an elementary submodel, κ is a cardinal with $\omega_1 \leq cof(\kappa) = \tau < \kappa$ and $\langle X, \mathcal{T}\rangle$ is a topological space such that $X, \mathcal{T}, \kappa \in M$ and $\tau \subseteq M$ but $\kappa \not\subseteq M$. If X_M is homeomorphic to κ then X is not homeomorphic to κ.*

Proof: Again, we have two cases . Let $w(X)$ be the least cardinal of a base for X. If $w(X) < \kappa$ or $w(X) > \kappa$ then clearly X is not homeomorphic to κ. We may suppose, then that $w(X) = \kappa$. Pick $\mathcal{B} = \{B_\beta : \beta < \kappa\} \in M$ a basis for X of size κ. Fix $f : \tau \longrightarrow \kappa$ a strictly increasing cofinal map and define for $x \in X \cap M$:

$$\forall \alpha < \tau,\ C_{x,\alpha} = \bigcap \{B_\beta : \beta < f(\alpha) \text{ and } x \in B_\beta\}$$

Notice that as all parameters are in M, also $C_{x,\alpha} \in M$.

Define:

$$P_x = \{C_{x,\alpha} : \alpha < \tau\}.$$

Notice that:

(1) $P_x \in M$.
(2) $P_x \subseteq M$ (since $\tau \subseteq M$).
(3) $\bigcap P_x = \{x\}$ since if $y \neq x$ then there is B_β such that $y \notin B_\beta$ but $x \in B_\beta$. We are using that X is T_1; this is by elementarity, since X_M is T_1 because κ is. Pick $\alpha \in \tau$ such that $f(\alpha) > \beta$ and notice that $y \notin C_{x,\alpha}$.

To finish the proof observe that, defining $\mathcal{V}_x$ and π as in 0.1,

$$\bigcap \mathcal{V}_x \subseteq \bigcap P_x = \{x\}.$$

So, as an injective perfect map, π is a homeomorphism, and the proof proceeds as in Theorem 1.8.

□

Lemma 1.10. [5] *If $0^{\#}$ does not exist and $|M| \geq \kappa$ then $\kappa \subseteq M$.*

This is proved by considering the inverse i^{-1} of the Mostowski collapsing isomorphism i of M. $i^{-1}\big|L_\kappa$ moves some $\alpha < \kappa$ if $M \not\supseteq \kappa$.

Theorem 1.7 together with the previous lemma leads to the result:

Corollary 1.11. *Suppose $0^{\#}$ does not exist. Let M be an elementary submodel, $\langle X, \mathcal{T} \rangle$ a topological space and κ a cardinal with $cof(\kappa) \geq \omega_1$ such that $\kappa, X, \mathcal{T} \in M$. Then if X_M is homeomorphic to κ, then X is homeomorphic to κ.*

The reader may obtain information about $0^{\#}$ in [4]. It is a special subset of ω. Its existence has many consequences, among them the existence of large cardinals in inner models. Also, $V = L \to 0^{\#}$ does not exist.

Any definable κ such as ω_1, ω_2, etc.. is automatically in M. Also, the condition that $\kappa \in M$ is sometimes a consequence of $\kappa \subseteq M$. This holds for instance for all successor cardinals:

Proposition 1.12. *Suppose that $\kappa = \aleph_\beta$ and $\beta < \kappa$. If M is an elementary submodel and $\kappa \subseteq M$ then $\kappa \in M$.*

Proof: The proof is simple. We have that $\kappa = \aleph_\beta$ and $\beta \in M$ so $\aleph_\beta$ is defined in M , so $\kappa \in M$.

□

We now present some examples. Let $ot(X)$ be the order type of X.

Lemma 1.13. *Let M be an elementary submodel and $\kappa \in M$ a cardinal with the order topology $\mathcal{T}$. Then κ_M is homeomorphic to $ot(\kappa \cap M)$.*

Proof: To see this, $\mathcal{B} = \{[\alpha, \beta], [\alpha, \beta) : \alpha < \beta < \kappa\}$ is a basis for the topology of κ that lies in M. Therefore $\mathcal{T}_M$ and $\mathcal{B}_M = \{B \cap M : B \in \mathcal{B} \cap M\}$ generate the same topology on κ_M.

Now $[\alpha, \beta) \in M$ if and only if $\alpha, \beta \in M$, and similarly for $[\alpha, \beta]$.

So if $f : ot(\kappa \cap M) \longrightarrow \kappa \cap M$ is strictly increasing and onto , it is a homeomorphism between $ot(\kappa \cap M)$ and κ_M.

□

Example 1.14. *An elementary submodel M , and a topological space $\langle X, \mathcal{T}\rangle \in M$ such that X is not homeomorphic to ω but X_M is homeomorphic to ω.*

Just pick X a discrete space of uncountable cardinality and M a countable elementary submodel such that $X \in M$. In this case X_M is a countable discrete space and thus homeomorphic to ω.
$\square$

Notice that in the above example, $\omega \in M,\ \omega \subseteq M$ but $cof(\omega) = \omega < \omega_1$.

If we are looking for an X such that X_M is homeomorphic to ω_1 but $X \neq X_M$, by Theorem 1.7 we need to find an M with $\omega_1 \not\subseteq M$. Such an M must have $|\omega_1 \cap M|$ countable and yet be uncountable since X_M is uncountable. We thus assume Chang's Conjecture in the following form:

There is an elementary submodel M such that $|M| = \aleph_1$, $|\omega_1 \cap M| = \aleph_0$ and $|\omega_2 \cap M| = \aleph_1$.

Example 1.15. *Assuming Chang's Conjecture, there is an elementary submodel M , and a topological space $\langle X, \mathcal{T}\rangle \in M$ such that X is not homeomorphic to ω_1 but X_M is homeomorphic to ω_1.*

Observe that:

$$\omega_2 = min\{\tau \in M : |\tau \cap M| = \aleph_1\}(*).$$

To see this, pick $\tau < \omega_2 \in M$. Then there is $f \in M$, a bijection between ω_1 and τ. Because $f \in M$, we have that $|\tau \cap M| = |\omega_1 \cap M|$.

Claim : The order type of $\omega_2 \cap M$ is ω_1.

To see that suppose otherwise that there is a $\beta > \omega_1$ and a $g : \beta \longrightarrow \omega_2 \cap M$ strictly increasing and onto. Now $f(\omega_1) < \omega_2$ would contradict $(*)$.

Let $X = \omega_2$ with the order topology. By Lemma 1.13, X_M is homeomorphic to ω_1.
$\square$

Chang's Conjecture has medium large cardinal strength - its consistency can be obtained from an ω_1-Erdös cardinal α, i.e. $\alpha \to (\omega_1)^{<\omega}$ [7].

2. The Long line

Theorem 2.1. *If X_M is homeomorphic to the long line, then $X = X_M$.*

Proof: X_M is locally compact T_2, so X is also by Lemma 1.4. It is easy to see that X is connected since X_M is. Since X is connected and Tychonoff, there is a continuous f mapping X onto $[0,1]$. Hence there is a continuous g mapping X_M onto $[0,1]_M = [0,1] \cap M$ by Lemma 1.6. But then $[0,1] \cap M$ is connected, it includes the rationals, so it is all of $[0,1]$.

We may therefore conclude by Lemma 1.5 that $\omega_1 \subseteq M$.

In [8] it is shown that $\omega_1 \subseteq M$ implies hereditary Lindelöfness goes up from X_M to X; the same proof will work for local hereditary Lindelöfness:

Lemma 2.2. *If $\omega_1 \subseteq M$ and X_M is locally hereditarily Lindelöf, so is X.*

Proof. Suppose X is not locally hereditarily Lindelöf. Then:
$M \models (\exists x \in X)(\forall U \in \mathcal{T})[x \in U \rightarrow (\exists f : \omega_1 \longrightarrow U)(\exists \{U_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{T})(\forall \alpha < \omega_1)\ (f(\alpha) \in U_\alpha$ and $(\forall \beta < \alpha)(f(\alpha) \notin U_\beta))]$.

Since $\omega_1 \subseteq M$, we have:

$(\exists x \in X \cap M)(\forall U \in \mathcal{T} \cap M)[x \in U \cap M \rightarrow (\exists f \in M)(f : \omega_1 \longrightarrow U \cap M)(\exists \{U_\alpha\}_{\alpha<\omega_1} \subseteq \mathcal{T} \cap M)(\forall \alpha < \omega_1)\ (f(\alpha) \in U_\alpha \cap M$ and $(\forall \beta < \alpha)(f(\alpha) \notin U_\beta \cap M))]$,

which implies X_M is not locally hereditarily Lindelöf.

□

It follows that:

Lemma 2.3. *X is first countable and hence X_M is a subspace of X.*

Proof. The second half follows from the first and Lemma 1.6; the first half follows from local compactness plus local hereditary Lindelöfness.

□

Now we can finish the proof of Theorem 2.1:

Define by induction $\{K_\alpha\}_{\alpha<\omega_1}$: K_0 is any compact neighborhood in X; for α limit, $K_\alpha = \overline{\bigcup\{K_\beta\}_{\beta<\alpha}}$; for $\alpha = \beta + 1$, take a compact neighborhood N_x about each x in K_β and let $K_\alpha = \overline{\bigcup\{N_x : x \in K_\beta\}}$.

Let $K = \bigcup_{\alpha<\omega_1} K_\alpha$. K is open by construction and closed by first countability. Therefore $K = X$ by connectedness. By induction, each K_α - and hence X - has cardinality $\leq 2^{\aleph_0}$, since compact first countable Hausdorff spaces have cardinality $\leq 2^{\aleph_0}$, and the closure in a first countable Hausdorff space of a set of size $\leq 2^{\aleph_0}$ also has size $\leq 2^{\aleph_0}$. But then $X \subseteq M$ by Lemma 1.5. X_M is a subspace of X, so the topology of X_M coincides with that of $X \cap M$, i.e. X, so we are done.

We have used very little of the properties of the long line; what we have in fact proved is:

Theorem 2.4. *If X_M is a connected, locally compact, locally hereditarily Lindelöf T_2 space, then $X = X_M$.*

Theorem 2.4 should be compared with the following result, which is Theorem 17 of [8]:

Theorem 2.5. *If X_M is a locally compact, hereditarily Lindelöf uncountable T_2 space, then $X = X_M$.*

"Connected" cannot be replaced by "locally connected" in Theorem 2.1: Let X be the disjoint sum of $(2^{\aleph_0})^+$ copies of $\mathbb{R}$, and let M be a countably closed elementary submodel of size $2^{\aleph_0}$. Then X_M is the sum of $2^{\aleph_0}$ copies of $\mathbb{R}$.

The local hereditary Lindelöfness also cannot be omitted. Consider a "longer line", obtained by ordering $X = (2^{\aleph_0})^+ \times \mathbb{R}$ lexicographically. Take a countably closed elementary submodel M of size $2^{\aleph_0}$. Then X_M is an initial segment of X and so is locally compact T_2 and connected.

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