j-3 Consistency Results in Topology, II: Forcing and Large Cardinals

When Quotable Principles fail one will have to turn to the machinery of consistency proofs itself for a solution of the problem.

Many topologists are familiar with the forcing method, which we will describe in the first section of this article. This method works fairly well when the problems involve sets of bounded cardinality but it tends to fail when one wants to prove something like "*all* spaces that are such and such are so and so". A prime example is the ***Normal Moore Space Conjecture***, which we now know cannot be proved consistent assuming the consistency of ZFC alone – the proof requires additional, stronger, axioms that most set theorists regard as safe to assume. Such axioms assert the (consistency of the) existence of **large cardinal**s; these will be described in the second section. In the last section we give a small sample of consistency results.

This article needs to be read in conjunction with the preceding one in this volume, in order to get a reasonable picture of the state of consistency results in topology. In particular, almost all of the propositions listed as consistent here are known to be undecidable, but the consistency of their negations is often not mentioned, since it is most easily derived from a combinatorial principle such as the ones discussed in the preceding article.

Note to the reader: there is an extended version of this article, with references for all the results mentioned and more, available on Topology Atlas, see [6].

1. Forcing

Forcing is a method for producing a new model of ZFC from a given one, called "the ground model". Proving that the method works requires much attention to metamathematical details (see [Ku], where an accessible introduction to the forcing method can be found). The actual applications of forcing, however, mainly boil down to verifying combinatorial properties of partial orders, orders either "taken off the shelf" or specially constructed for the problem at hand. Forcing is used to prove the consistency of both existential and universal propositions.

To prove the consistency of an existential proposition one constructs a partial order consisting of approximations to the desired object, with the approximations (which are usually smaller in size than the desired object) being ordered by $p \leqslant q$ if p is a "better" approximation than q – this is like the discussion of MA in the previous article but with a major difference, as will become clear momentarily. For example, to prove it consistent that $2^{\aleph_0} \geqslant \aleph_2$, one could consider finite approximations to a listing of $\aleph_2$ distinct functions from ω into 2, i.e., finite partial functions from $\omega_2 \times \omega$ into 2. These approximations (forcing conditions) are then ordered by extension. A filter (i.e., an upward closed consistent subset) for that partial order $\mathcal{P}$ naturally yields a partial function from $\omega_2 \times \omega$ into 2. The trick is to make that function total. For that to happen, it would suffice to show that for each $\alpha \in \omega_2$ and $n \in \omega$, the filter met

$$D_{\alpha,n} = \{p \in P\colon \langle\alpha, n\rangle \in \operatorname{dom} p\}.$$

Note $D_{\alpha,n}$ is a **dense set**, i.e., every element of the partial order has an extension in $D_{\alpha,n}$.

Given any countable collection of dense sets, it is easy to construct a filter that meets each of them; a metamathematical argument establishes that in fact when forcing, one only has to consider those dense subsets of the partial order which lie in a fixed countable model M. Thus one gets a filter that simultaneously meets all the $D_{\alpha,n}$'s in M and thus gets the *generic* function to be total. The requirement that the induced map from ω_2 to $\mathcal{P}(\omega)$ be one-to-one can actually be dispensed with by meeting those $D_{\alpha,\beta}$'s, $\alpha, \beta \in \omega_2$, which are in M, where

$$D_{\alpha,\beta} = \{p \in P\colon (\exists n)\big(p(\alpha, n) \neq p(\beta, n)\big)\}.$$

This shows the difference with the MA approach: there the filter meeting the 'good' dense sets belongs to M, in the present example, if CH holds in M there is no filter in M that meets all $D_{\alpha,\beta}$'s; we adjoin the filter to M and use it to construct an extension $M[G]$. This extension is a model of ZFC and in it there is an injective map from ω_2 into $\mathcal{P}(\omega)$. If the ground model M satisfies the GCH and we use ω_3 instead of ω_2 then in $M[G]$ all almost disjoint families of uncountable subsets of ω_1 have cardinality at most $\aleph_2$, which is strictly less than $2^{\aleph_0} = 2^{\aleph_1}$.

A ***Souslin tree*** can be constructed by ***σ-closed*** or ccc forcing. The existence of a Souslin tree shows that $\mathsf{MA} + \neg\mathsf{CH}$ fails but one can do much better: Baumgartner constructed a model where $\mathsf{MA}(\aleph_1)$ fails completely in that for *every* ccc partial order there is a family of $\aleph_1$ many dense sets such that *no* filter meets them all.

To prove the consistency of a universal proposition, one usually uses **iterated forcing** (or repeated forcing), see [Ku] and [1]. For example, to prove the consistency of "all perfectly marvelous subsets of $\mathbb{R}^2$ are splendiferous", one would iterate the process of forcing one perfectly marvelous subset to be splendiferous. Alternatively, one could iteratively force to "kill" non-splendiferous perfectly marvelous subsets. Of course, inevitable difficulties arise. The process of iterating the forcing could introduce new perfectly marvelous non-splendiferous sets, so one has to arrange that the new ones are either made splendiferous or are killed, i.e., made not perfectly marvelous. This is done by an "*initial stage*" argument – e.g., if the Continuum Hypothesis is assumed, subsets of $\mathbb{R}^2$ have cardinality $\aleph_1$; if one iterates $\aleph_2$ times, one argues that every perfectly marvelous non-splendiferous subset appears at some stage and is taken care of there. One also needs, e.g., that perfectly marvelous sets one has forced to make splendiferous stay that way. Thus one needs "preservation arguments".

A typical example of a universal proposition proved consistent by iterated forcing is the ***Souslin hypothesis***. Here we take as our partial orders the Souslin trees themselves – old and new – with the order being the reverse of the tree order. Actually, we first prune the trees so that each element of the tree has successors at all levels beyond it. Then, noting that compatibility $=$ comparability in a tree, we observe that a filter meeting each

$$D_\alpha = \{t\colon \text{ the height of } t \text{ is at least } \alpha\}$$

is a ***branch*** of length ω_1 and therefore destroys Souslinity. Once a tree has a branch all the way through it, it stays that way, so one iterates $\aleph_2$ times; the Souslin trees have cardinality $\aleph_1$ so appear at an initial stage, and, when killed, stay dead.

This proof and others like it gave us both the formulation and consistency proof of $\mathsf{MA} + \neg\mathsf{CH}$; a study of this proof, as presented in [Ku] for example, will reveal the basic technical problems one usually encounters in iterated forcing constructions.

A complication that arises in dealing with topological spaces rather than with algebraic objects is that, after forcing to produce a new (larger) model of set theory (called the **forcing extension**, the **generic extension** or just the **extension**), a topological space in the ground model is no longer a topological space; the best one can do is use the former topology as a basis for a topology on the original set. This preserves the separation axioms up to ***complete regularity*** but properties like normality can be destroyed or created, see [4].

2. Large cardinals

Although set theorists have investigated a plethora of large cardinals, we will confine ourselves here to several that appear most often in topological contexts, namely inaccessible, weakly compact, measurable, strongly compact, supercompact, and huge ones. In the definitions we shall implicitly assume all the large cardinals are uncountable.

A note on the term **large cardinal** is in order. A cardinal number is 'large' if the assumption of its existence, when added to the axioms of ZFC, proves the consistency of ZFC. This works as follows, for any cardinal κ one can consider the set H_κ – the set of all sets which have size less than κ and whose members and members of members and … all have size less than κ. Loosely speaking κ is large if H_κ is a model of ZFC.

A cardinal number κ is an **inaccessible cardinal** (also a **strongly inaccessible cardinal**) if it is regular and $2^\lambda < \kappa$ whenever $\lambda < \kappa$ is a cardinal. κ is a **weakly compact cardinal** if it is inaccessible and, whenever T is a tree of height κ with levels of size less than κ, then T has a branch of length κ. κ is a **measurable cardinal** if there is a non-principal **κ-complete ultrafilter** (i.e., closed under intersections of size less than κ). A cardinal κ is a **strongly compact cardinal** if every κ-complete filter can be extended to a κ-complete ultrafilter.

Sometimes measurability is defined using countable completeness (i.e., ω_1-completeness) rather than κ-completeness. Let us call such cardinals **Ulam measurable**. The least Ulam-measurable cardinal is in fact measurable.

All of these cardinals have several equivalent formulations. The easiest to state are often in terms of ultrafilters, but the most useful involve elementary embeddings:

An **inner model** is a class $M = \{x\colon \varphi(x)\}$, for some formula φ, such that ZFC holds in M. An **elementary embedding** $j : V \to M$, where V is the universe of sets, is a function such that for every $a_1, \dots, a_n \in V$, and for every formula $\psi(x_1, \dots, x_n)$, $\psi(a_1, \dots, a_n)$ holds if and only if $\psi\big(j(a_1), \dots, j(a_n)\big)$ holds in M. M is **closed under λ-sequences** if ${}^\lambda M$, the class of all λ-sequences of members of M, is a subclass of M.

One can prove that κ is measurable if and only if there is an inner model M closed under κ-sequences and an elementary embedding $j : V \to M$ such that $j(\kappa) > \kappa$.

We now define supercompactness as a stronger version of measurability: κ is a **supercompact cardinal** if for every $\lambda \geqslant \kappa$, there is an inner model M_λ closed under λ-sequences, and an elementary embedding $j_\lambda : V \to M_\lambda$, such that $j_\lambda(\kappa) > \lambda$.

As for measurability, there is an equivalent formulation, which we omit, which avoids the apparent difficulty of quantifying over formulas.

I have listed these cardinals in order of increasing strength, i.e., every supercompact cardinal is strongly compact, every measurable cardinal is weakly compact, and so forth. Finer analyses of large cardinals consider a hierarchy of **consistency strength**, i.e., the consistency of a cardinal with property P implies the consistency of a cardinal with property Q. For example, define huge cardinals as another generalization of measurable ones.

A cardinal κ is a **huge cardinal** if there is an inner model M and elementary embedding $j : V \to M$ such that $j(\kappa) > \kappa$ and M is closed under $j(\kappa)$-sequences.

Huge cardinals need not be supercompact, but their consistency strength is strictly stronger than supercompactness. For more information about large cardinals, see Kanamori's book [5]. Large cardinals may either be used directly in a proof or may be used to construct a model of set theory in which a desired proposition holds.

Very often a large cardinal property is a generalization of a property of $\aleph_0$ to the uncountable. For instance, $\aleph_0$ is clearly inaccessible: it is regular and $2^n < \aleph_0$ for all $n \in \omega$. For another example, consider trees. A κ-**Aronszajn tree** is a tree of height κ whose levels are of cardinality less than κ and with no κ-branch. For definitions, see Todorčević's survey [KV, Chapter 6]. **König's Tree Lemma** says there is no $\aleph_0$-Aronszajn tree. Generalizing this plus inaccessibility gives us weak compactness. Likewise the existence of ultrafilters on ω says that $\aleph_0$ is 'measurable'.

Large cardinals sometimes appear in purely topological circumstances. We know from ***Jones' Lemma*** that $2^{|D|} \leqslant 2^{d(X)}$ whenever D is a closed discrete subset of a normal space X, where $d(X)$ denotes the ***density*** of X. The ***extent*** of X, denoted $e(X)$, is the supremum of the cardinalities of the closed discrete subsets of X and this suggests the natural question whether also $2^{e(X)} \leqslant 2^{d(X)}$ for normal spaces. This leads to inaccessible cardinals: if $2^{e(X)} > 2^{d(X)}$ then $e(X)$ is a **weakly inaccessible cardinal** (a regular limit cardinal) and from an inaccessible cardinal one can prove the consistency of the existence of a normal space satisfying the above inequality.

An inaccessible cardinal is weakly compact if and only if its ***absolute*** is normal – here the cardinal carries the ***order topology***.

Measurable cardinals date back to the 1930s and have a number of significant – although isolated – direct applications to general topology. For example, a discrete space of size κ is ***realcompact*** if and only κ is not Ulam-measurable. It is also easy to prove that if X is a Lindelöf space with points G_δ, then $|X|$ is less than the first measurable cardinal. On a different tack the existence of a measurable cardinal is equiconsistent with the existence of a ***Baire space*** without isolated points which is **irresolvable**, i.e., any two dense sets meet.

Strongly compact and weakly compact cardinals can be equivalently formulated topologically: κ is strongly compact if and only if the κ**-box product** of κ**-compact space**s is κ-compact, wherein one takes the ***Tychonoff Product Theorem*** and replaces "finite" by "$< \kappa$" everywhere; κ is weakly compact is the ordinary product of κ-compact spaces is again κ-compact.

There are other straightforward applications, e.g., if κ is weakly compact and X is $< \kappa$-collectionwise Hausdorff and $\chi(X) < \kappa$, then X is κ-collectionwise Hausdorff. A more difficult direct application is due to Watson, who proved that if there is a strongly compact cardinal, there is a σ***-discrete*** hereditarily normal ***Dowker space***, see the corresponding article in this volume.

The most significant uses of large cardinals in topology occur in contexts in which one is proving the consistency of universal statements about objects of unbounded cardinality, for example, the **Normal Moore Space Conjecture**: all normal Moore spaces are metrizable, or the **Moore–Mrówka problem**: compact spaces of countable tightness are sequential. The latter is an application of the ***Proper Forcing Axiom*** (see the previous article), which is proved consistent from the consistency of a supercompact cardinal, and applications of which – in contrast to those of Martin's Axiom – often require the practitioner to actually do some forcing. Frequently, finer analyses of PFA consequences reveal that in fact one need only consider objects of bounded cardinality, in particular $\aleph_1$. In such cases, a more delicate forcing argument enables one to avoid large cardinals. That is the case with the Moore–Mrówka problem referred to above.

The Normal Moore Space Conjecture – more generally the assertion that normal spaces of character less than the continuum are collectionwise normal – was first shown consistent by Nyikos who derived it in ZFC from the ***Product Measure Extension Axiom***, which had been proved consistent by Kunen from the existence of a strongly compact cardinal (see [KV, Chapter 16]). More general results that do not depend on measures were established in [3], wherein a general framework was set up for proving the consistency of universal topological assertions involving objects of unbounded cardinality and spaces of small character from the consistency of supercompact (in special cases, strongly compact) cardinals. Applying the machine to, e.g., getting a model in which all silly spaces of character $< \kappa$ (the supercompact cardinal which will become small, e.g., $2^{\aleph_0}$) are ridiculous will reduce the problem to showing an appropriate forcing notion **preserves** non-ridiculousness. When we say a forcing *preserves* a topological property, we mean that if a space satisfies the property in the ground model, then the space it generates in the extension satisfies the property there.

Balogh noticed that in the particular context of normality vs. collectionwise normality, this framework could be modified, weakening the character restriction, in order to obtain that if it is consistent that there is a supercompact cardinal, it is consistent that all normal spaces of ***pointwise countable type*** – in particular all locally compact normal spaces and all first-countable spaces – are collectionwise normal; a space is of **pointwise countable type** (or of **point-countable type**) if every point is contained in a compact set with a countable neighbourhood base (it is of **countable type** if every compact set is contained in such a compact set). Grunberg, Junqueira and Tall [4] then extended the method of [3] to obtain a general method for proving the consistency of universal topological assertions involving objects of unbounded cardinality and spaces of small pointwise type, obtaining in particular a more useful proof of Balogh's result.

Supercompact cardinals are the most useful of large cardinals for topologists, because they yield the Proper Forcing Axiom, and, in contrast to weakly compact, measurable, and huge cardinals, not only affect the cardinal itself, but all larger cardinals. This phenomena is referred to as **reflection**. Roughly speaking it tells us that if there is a counterexample to some universal statement φ, there is one of size less than the supercompact cardinal. For particular φ, one can then try to make the supercompact cardinal κ small (e.g., $\aleph_1$, $\aleph_2$ or $2^{\aleph_0}$) by forcing, and perhaps be able to prove that the reflection phenomena for φ at κ still hold.

In a similar vein one can take a property holding at a large cardinal and bring it down so that some small cardinal such

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as $\aleph_1$, $\aleph_2$, or $2^{\aleph_0}$ has that property. In this case – where one is not so interested in what happens for cardinals larger than that particular small cardinal – one can usually make do with a weakly compact or measurable cardinal rather than a supercompact cardinal. For example, a weakly compact cardinal may be collapsed to $\aleph_2$ to create a model in which there are no $\aleph_2$-Aronszajn trees, see [KV, Chapter 6]. A more topological example is Shelah's theorem that if it is consistent that there is a weakly compact cardinal then it is consistent that there are no Lindelöf spaces of size $\aleph_2$ with all points G_δ, which brings down the fact there is no Lindelöf space of size κ with all points G_δ, if κ is weakly compact.

The first use of supercompact cardinals in topology was due to Shelah who proved that if it is consistent there is a supercompact cardinal, then it is consistent that locally separable first-countable $\aleph_1$-collectionwise Hausdorff spaces are collectionwise Hausdorff. The proof (or its progenitor concerning reflection of ***stationary sets***) is a prototypical argument; other applications include Axiom R and its consequences, see [2] and [HvM, Chapter 1].

Whenever one uses large cardinals to establish (the consistency of) a topological statement, one wonders whether they are actually necessary. As mentioned earlier, consequences proved from PFA can often be proved consistent without large cardinals. The usual way one shows that a statement φ requires large cardinals for its proof is to show that if φ holds, then there is an inner model which has a large cardinal. For example, having noted [Ku, VIII, §3] that the consistency of the existence of an inaccessible cardinal enables one to prove the consistency of there being no Kurepa trees, one then shows that if there is no ***Kurepa tree***, then $\aleph_2$ is an inaccessible cardinal in Gödel's constructible universe L (see [Ku, VII, B9]), and hence that it is consistent that there is an inaccessible cardinal.

L yis an **inner model for inaccessible cardinals**, which means that if κ is an inaccessible cardinal, then κ is inaccessible in L. There are more complicated inner models for a measurable cardinal, for "many" measurable cardinals (e.g., if there is a sequence of measurable cardinals, they are all still measurable in the inner model), etc. These inner models however do have L-like characteristics which are useful in showing that large cardinals are required in order to obtain the consistency of certain propositions. Finding inner models for supercompact cardinals is an ongoing area of research in set theory; at present there are no good techniques for showing that the use of a supercompact cardinal in a consistency proof is necessary. In practice, topologists have not as yet actually engaged in inner model theory, but rather have shown that topological statements imply combinatorial statements of known large cardinal strength. A typical example is Fleissner's proof (see [KV, Chapter 16]) that the Normal Moore Space Conjecture has large cardinal strength. He proved that the NMSC entailed the failure of the **Covering Lemma**. The Covering Lemma (for an inner model M) asserts that every uncountable set is included in some member of M of the same cardinality. The failure of the Covering Lemma for, e.g., M, an inner model for "many" measurables implies that in fact there are "many" measurables in M.

Another example of this technique is due to C. Good, who showed that the Covering Lemma entails the existence of *a first-countable Dowker space* with sundry additional properties (see the article on Dowker spaces). Note: the Covering Lemma for L is often referred to via an equivalent formulation as "$0^\#$ does not exist".

3. Methods and models

In this section we shall briefly discuss some of the most useful models employed in consistency results, equivalently, the most useful partial orders.

Cohen reals

The simplest non-trivial forcing employs the partial order of finite partial functions from ω into 2 ordered by reverse inclusion. It adds a new $f : \omega \to 2$ and hence a new real, called a **Cohen real**. Interestingly, from a single Cohen real one can already construct a Souslin tree.

Using finite partial functions from $\kappa \times \omega$ into 2 instead, as mentioned previously, one shows that κ new reals are added. Since $|\kappa| = |\kappa \times \omega|$, one may as well use κ instead of $\kappa \times \omega$. This forcing is referred to as "adding κ Cohen reals" and the corresponding extension is commonly called the **Cohen model**. It is particularly useful when κ is supercompact since reflection phenomena persist. For example, every space of character $< 2^{\aleph_0}$ in which subspaces of size $< 2^{\aleph_0}$ are metrizable is metrizable [3]. Aside from general machinery for handling large cardinals and forcing, the key lemma is that adding Cohen reals preserves non-metrizability. Typically, one gets some weaker results without large cardinals, e.g., upon adding $\aleph_2$ Cohen reals, if a first-countable space of weight $\aleph_1$ has all its subspaces of size $\aleph_1$ metrizable, then it is metrizable.

Countably closed forcing

Countably closed forcing (every countable descending sequence of conditions has a lower bound) covers a wide variety of models, so it is somewhat misleading to group them together since they may exhibit incompatible behaviors. Nonetheless they do have common features such as L-like phenomena (e.g., $\diamondsuit$ is forced if new subsets of ω_1 are added) and the use of an ω_1-descending sequence of forcing conditions deciding the properties of a function from ω_1 into V in the extension. Most of the time one is interested in such orders with countable conditions. The simplest example is forcing with countable partial functions from κ into 2, where κ is a regular cardinal. For $\kappa \geqslant \aleph_2$, this yields a model in which normal spaces of character $< \kappa$ are $\aleph_1$-collectionwise Hausdorff. This forcing is called "adding κ Cohen subsets of ω_1".

A very useful one alluded to earlier is the Lévy-collapse, which makes a large cardinal κ into $\aleph_2$ by creating maps from ω_1 onto each uncountable cardinal smaller than κ. Countably closed partial orders using countable conditions to create a subset of ω_2 have been employed to construct complicated Lindelöf spaces, e.g., ones of size $\aleph_2$ with points G_δ or ones of that size with no Lindelöf subspaces of size $\aleph_1$.

Random real forcing

This is forcing with the product measure algebra on $\{0,1\}^\kappa$. We say we are "adding κ random reals". See [KV, Chapter 20] for a description. The new **random real**s are obtained from a generic filter G as follows: let F denote the set of finite partial functions from κ to 2 whose associates clopen subset of $\{0,1\}^\kappa$ belongs to G. The union $\bigcup F$ is a total function from κ to 2 and gives rise to new reals via a bijection between κ and $\kappa \times \omega$.

Adding at least $\aleph_2$ random reals yields a model in which $\mathfrak{d} = \aleph_1$ yet $2^{\aleph_0} > \aleph_1$, see the article on $\beta\mathbb{N}$ for the definition of $\mathfrak{d}$. It is particularly interesting to force to add $\aleph_2$ or more random reals over a model of $\mathsf{MA} + \neg\mathsf{CH}$, since much of that axiom is preserved. For instance, in contrast with just one Cohen real, no Souslin tree is created. Adding strongly compact many random reals produces a model of the Product Measure Extension Axiom mentioned earlier, see [KV, Chapter 16].

Other forcings

There are many other ways of adding new reals; the 'total failure of MA' alluded to above was obtained by adding Sacks reals.

The Proper Forcing Axiom, has many strong consequences mentioned above. To prove a proposition φ is undecidable by $\mathsf{MA} + \neg\mathsf{CH}$, the standard stratagem is to show PFA implies φ, but that a carefully constructed example of $\neg\varphi$ constructed from CH or $\diamondsuit$ remains an example under countable chain condition forcing. A noteworthy example is Szentmiklóssy's ***S-space*** that is consistent with $\mathsf{MA} + \neg\mathsf{CH}$. Another one is a Baire space of size $\aleph_1$ without isolated points, consistent with $\mathsf{MA} + \neg\mathsf{CH}$.

It is often useful to have some example that contradicts $\mathsf{MA} + \neg\mathsf{CH}$, while retaining as much of that axiom as possible. The standard procedure is to construct the example using CH or $\diamondsuit$ and then iterate forcing with partial orders having a strong form of countable chain condition that preserves the example. The first such example was a Souslin line consistent with Martin's Axiom for partial orders with **property** K (every uncountable set contains an uncountable subset of pairwise compatible elements).

Consistency results in topology have come a long way from the initial applications of $\mathsf{MA} + \neg\mathsf{CH}$ and $V = L$. A recent trend in particular is to obtain models in which consequences of these two contradictory axioms hold simultaneously. The key idea is to first force to construct a particularly nice ***Souslin tree*** S, then force as much of $\mathsf{MA} + \neg\mathsf{CH}$ as can possibly be compatible with the existence of S, and then force with S. This stratagem enabled Larson and Todorčević to solve an old problem of Katětov by showing it consistent that whenever X^2 is compact hereditarily normal, then X is metrizable. In this model there are no S- or L-subspaces of compact first-countable spaces – a strong consequence of $\mathsf{MA} + \neg\mathsf{CH}$, yet there are also no ***Q-set***s, contradicting $\mathsf{MA} + \neg\mathsf{CH}$.

Extending the method to PFA, rather than MA, yields that subspaces of compact spaces with countable tightness are hereditarily Lindelöf if and only if they are hereditarily separable (Todorčević). By doing preliminary forcing one can also get normal first-countable spaces to be collectionwise Hausdorff. This was used by Larson and Tall to obtain the consistency (relative to a supercompact cardinal) of every locally compact perfectly normal space being paracompact.

By using proper partial orders that don't add reals, one can get some consequences of PFA consistent with CH. For example, that every compact space of countable tightness is sequentially compact and has points of character $\leqslant \aleph_1$. It is possible to obtain some of the consequences of PFA that imply $\neg$CH and yet still have $2^{\aleph_0} < 2^{\aleph_1}$. This was accomplished by Eisworth, Nyikos and Shelah in order to obtain the consistency of there being no separable, hereditarily normal, locally compact space of size $\aleph_1$.

As with any other field of mathematics that has reached a certain level of maturity, particular difficult problems require either a new method apparently unique to them or an intricate combination of known methods. An example of the latter is the difficult forcing construction by Gruenhage and Koszmider of a locally compact normal metacompact space which is not paracompact.

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