

LOCALLY COMPACT PERFECTLY NORMAL SPACES

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ABSTRACT. Using results announced by S. Todorćevic we establish that if it is consistent that there is a supercompact cardinal then it is consistent that every locally compact perfectly normal space is paracompact. Modulo the large cardinal, this answers a question of S. Watson. We also solve a problem raised by the second author, proving that it is consistent with ZFC that every first countable hereditarily normal countable chain condition space is hereditarily separable. Finally, we show that if it is consistent that there is a supercompact cardinal, it is consistent that every locally compact space with a hereditarily normal square is metrizable.

0. INTRODUCTION

Only a few of the implications concerning basic properties in general topology have remained open. One raised by Watson [Wa] is particularly interesting and is characterized there as his favorite problem:

Is it consistent that every locally compact perfectly normal space is paracompact?

If so, locally compact, perfectly normal spaces have a very simple structure; they are simply the topological sum of σ -compact, perfectly normal – hence hereditarily Lindelöf and first countable – spaces. In fact, as we shall see, these pieces may be taken to be hereditarily separable as well.

Continuing the theme of “niceness,” let us note that many of the notorious counterexamples of set-theoretic topology are ruled out: every perfectly normal manifold is metrizable, every locally compact normal Moore space is metrizable, there are no Ostaszewski spaces and so forth.

A tantalizing aspect of the problem is that a positive solution follows from the conjunction of two statements known to follow from well-known but mutually inconsistent axioms. Specifically,

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- a) $V = L$ implies locally compact perfectly normal spaces are collectionwise Hausdorff.
- b) $MA + \sim CH$ implies locally compact perfectly normal collectionwise Hausdorff spaces are paracompact.

We shall show that, assuming the existence of large cardinals, there is a model in which both conclusions hold, answering Watson’s question. Precisely,

Theorem 1. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact perfectly normal space is paracompact.*

Variations of the proof of Theorem 1 will also solve two other interesting problems. In *Open Problems in Topology*, the second author [Ta3] asked whether it is consistent that every first countable hereditarily normal countable chain condition space is hereditarily separable. The attraction again was that this followed from consequences of $MA + \sim CH$ and $V = L$. We have:

Theorem 2. *It is consistent with ZFC that every first countable hereditarily normal countable chain condition space is hereditarily separable.*

In [LT2] an old problem of Katětov [Ka] was solved by establishing the consistency of *every compact space with a hereditarily normal square is metrizable*. At the cost of a supercompact cardinal, we extend this to locally compact spaces.

Theorem 3. *If it is consistent that there is a supercompact cardinal, it is consistent that every locally compact space with a hereditarily normal square is metrizable.*

1. NOTATION

Our set-theoretic notation is standard, as in [Ku]. Topological notation is from Engelking [En]. Since we mainly deal with locally compact spaces, it is convenient to assume all spaces are Hausdorff unless otherwise noted. However, note that the various results quoted about normality implying collectionwise Hausdorffness do not in fact require the assumption of Hausdorffness.

If S is a tree and α is an ordinal, we let $S(\alpha)$ denote the α th level of S .

2. WATSON’S PROBLEM

The context we shall consider is in the same family as that used to prove the consistency of the positive solution to Katětov’s problem [LT2]. This approach will surely find increasing use in set-theoretic topology since it produces strong “Suslin-type” [KuTa] consequences of $MA + \sim CH$, e.g. all Aronszajn trees are special, subspaces of countably tight compact spaces are hereditarily Lindelöf if and only if they are hereditarily separable, as well as – in the model we produce here – the important consequence of $V = L$ that all normal first countable spaces are collectionwise Hausdorff. These models are all obtained by starting with a model in which there is a *coherent Suslin tree*. This is a Suslin tree $S \subseteq \omega^{<\omega_1}$, closed

under finite modifications, such that $\{\alpha \in \text{dom}(s) \cap \text{dom}(t) : s(\alpha) \neq t(\alpha)\}$ is finite for all $s, t \in S$. The existence of such a tree follows from \diamond [Lar, SZ] and holds after adding one Cohen real [SZ]. Once one has such an S , one then forces the maximal amount of some forcing axiom such as MA_{ω_1} or PFA compatible with the existence of S . Then one forces with S . The details of how to do the penultimate forcing can be found in [F, Lar, Mi, Mi2]. Here we only need to know that these are iterations like those to establish MA_{ω_1} or PFA, but that certain posets are omitted. For various propositions ϕ , the proof that MA_{ω_1} or PFA implies ϕ can be modified up to prove that the weaker version of MA_{ω_1} or PFA implies S cannot force ϕ to fail. The ϕ in our case will comprise several propositions that together imply locally compact perfectly normal collectionwise Hausdorff spaces are paracompact. In addition, we either start from L (if we do not require large cardinals so as to obtain as much of PFA as possible) or else a certain Easton model, and observe that the iteration plus the Suslin forcing will not destroy the fact that normal first countable \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff. The final step is to show that forcing with S establishes that normal first countable spaces are \aleph_1 -collectionwise Hausdorff.

We shall first aim to produce the set theory needed to get that locally compact perfectly normal collectionwise Hausdorff spaces are paracompact, and then eliminate the collectionwise Hausdorff hypothesis as previously indicated. We shall need to use some results of Todorćevic [To2]. For S a coherent Suslin tree, consider the axioms:

$\text{MA}_{\omega_1}(\mathbf{S})$: There exists a coherent Suslin tree S , and if P is a partial order satisfying the countable chain condition which doesn't force an uncountable antichain in S , and D_ξ ($\xi < \omega_1$) is a sequence of dense open subsets of P , then there is a filter $G \subseteq P$ such that $G \cap D_\xi \neq \emptyset$ for each $\xi < \omega_1$.

$\text{PFA}(\mathbf{S})$: There exists a coherent Suslin tree S , and if P is a proper partial order which doesn't force an uncountable antichain in S , and D_ξ ($\xi < \omega_1$) is a sequence of dense open subsets of P , then there is a filter $G \subseteq P$ such that $G \cap D_\xi \neq \emptyset$ for each $\xi < \omega_1$.

The consistency of $\text{MA}_{\omega_1}(\mathbf{S})$ is established explicitly in [Lar], though very similar constructions had been studied earlier (in [F], for instance). The consistency of $\text{PFS}(\mathbf{S})$ (minus the coherence requirement, which presents no additional difficulties), was established in [M].

Todorćevic introduces the following notion:

Definition. Let X be a compact space and H a collection of non-principal ultrafilters on ω . X is said to be **H -accessible** if whenever $A \subseteq X$ and $x \in \overline{A}$, there is a sequence $\{a_n\}_{n < \omega}$ from A and an ultrafilter $h \in H$ such that $\mathbf{a}_n \xrightarrow{h} \mathbf{x}$, by which we mean that for every open U containing x , $\{n : a_n \in U\} \in h$. (We say x is the **h -limit** of $\{a_n\}_{n < \omega}$).

Recall that a space X is *countably tight* if whenever $y \in \overline{Y} \subseteq X$, there is a countable $Z \subseteq Y$ such that $y \in \overline{Z}$. Todorcevic then proves:

Lemma 5. (PFA(S)) *If \dot{K} is an S -name for a compact countably tight space, then K is S -forced to be accessible via the collection H of ground model ultrafilters.*

Recall that a subspace Y of a space X is *locally countable* if for each $y \in Y$ there is an open U_y about y containing only countably many members of Y . Y is *σ -discrete* if it is the union of countably many discrete subspaces. Todorcevic then proves:

Theorem 6. (PFA(S)) *If \dot{K} is an S -name of a compact space accessible via the collection of ground model ultrafilters, then S forces that every locally countable subset of K of size \aleph_1 is σ -discrete.*

∴ From this he gets:

Theorem 7. (PFA(S)) *If \dot{K} is an S -name for a compact space, then \dot{Y} is S -forced to be a hereditarily separable subspace of \dot{K} if and only if \dot{Y} is S -forced to be hereditarily Lindelöf.*

Let us also note the following fact which had been established earlier [LT2]:

Lemma 8. (MA_{ω_1} (S)) *If \dot{X} is an S -name for a first countable hereditarily Lindelöf space, then \dot{X} is S -forced to be hereditarily separable.*

We will use Lemma 5 to get that the one-point compactification of a locally compact perfectly normal space X is a space to which Theorem 6 can be applied. Standard techniques and Theorem 7 will then yield that X is paracompact if it is collectionwise Hausdorff, so let us establish the theorems on that subject that we need.

Let us recall some standard facts about “normality versus collectionwise normality” [Ta1].

Definition. Let κ be an infinite cardinal. A topological space is **κ -collectionwise Hausdorff** (**$<\kappa$ -collectionwise Hausdorff**) if each closed discrete subspace D of size $\leq \kappa$ ($< \kappa$) can be **separated**, i.e., there exist disjoint open sets $\{U_d\}_{d \in D}$ such that $d \in U_d$. A space is **collectionwise Hausdorff** if it is κ -collectionwise Hausdorff for every κ .

Definition [F1]. Let λ be a regular uncountable cardinal. $\mathcal{A} = \{A_f : f \in {}^\lambda \lambda\}$ is a **stationary system** for λ if each A_f is a stationary subset of λ , and whenever $\alpha \in \lambda$ and $f, g \in {}^\lambda \lambda$, if $f|_\alpha = g|_\alpha$ then $A_f \cap (\alpha+1) = A_g \cap (\alpha+1)$. \diamond **for stationary systems** (at λ) is the assertion that for each stationary system \mathcal{A} for λ , there is a sequence $\{f_\alpha\}_{\alpha < \lambda}$ such that $f_\alpha \in {}^\alpha \alpha$ and for each $f \in {}^\lambda \lambda$ there is a stationary $S \subseteq A_f$ such that $\beta \in S$ implies $f|_\beta = f_\beta$.

Fleissner proved:

Lemma 9. *Suppose κ is a regular uncountable cardinal, GCH holds at κ and above, and \diamond for stationary systems holds for all regular $\lambda \geq \kappa$. Then if X is a normal first countable $< \kappa$ -collectionwise Hausdorff space, then X is collectionwise Hausdorff.*

He also probably noticed the following results but the only reference for them we know of is [T₂].

Lemma 10. *Suppose λ is a regular uncountable cardinal. Adjoin λ^+ Cohen subsets of λ . Then \diamond for stationary systems holds at λ .*

Lemma 11. *Suppose \diamond for stationary systems holds at the regular uncountable cardinal λ . Force with a λ -chain condition partial order of size $\leq \lambda$. Then \diamond for stationary systems still holds at λ .*

Using these lemmas, it is not difficult to get that normal first countable spaces which are \aleph_1 -collectionwise Hausdorff will be collectionwise Hausdorff in the model obtained by S -forcing over a model of PFA(S), provided we start with an appropriate model over which to do the PFA(S) iteration. In particular, start with a model in which there is a supercompact cardinal κ . To simplify matters, we could establish GCH below κ by a “mild” forcing [K] keeping κ supercompact. We then make κ indestructible under κ -directed-closed forcing [Lav] and then Easton-force to add λ^+ Cohen subsets of λ for every regular cardinal $\lambda \geq \kappa$ [E]. This will establish GCH as well as \diamond for stationary systems for regular $\lambda \geq \kappa$, while keeping κ supercompact. We then force to create then, then force PFA(S) and then force with S . The iteration of these three forcings has the κ -chain condition and is of size κ , so we have established that normal first countable spaces that are $< \kappa$ -collectionwise Hausdorff are collectionwise Hausdorff. It is clear that the straightforward iteration to produce PFA(S) – if it works at all – will produce a model in which $\kappa = \aleph_2$, but in fact Farah [F] proves PFA(S) implies OCA, while PFA(S) implies MA(σ -centred) because σ -centred forcing doesn’t add uncountable chains to Suslin trees [KT]. It follows (see [Be]) that

Lemma 12. *PFA(S) implies $2^{\aleph_0} = \aleph_2$ and hence implies S forces $2^{\aleph_0} = \aleph_2$.*

The “hence” is because S -forcing satisfies the countable chain condition. We have then that

Lemma 13. *In the model obtained as above, normal first countable \aleph_1 -collectionwise Hausdorff spaces are collectionwise Hausdorff.*

It remains to prove normal first countable spaces are \aleph_1 -collectionwise Hausdorff in this model. In order to do that, we prove a purely set-theoretic combinatorial lemma:

Lemma 14. *After forcing with a Suslin tree, the following holds. Suppose that $\{N(\alpha, i) : i < \omega, \alpha < \omega_1\}$ are sets such that for all α, i , $S_{i+1}^\alpha \subseteq S_i^\alpha$. Suppose further that:*

For all $A \subseteq \omega_1$, there is an $f : \omega_1 \rightarrow \omega$ such that

$$\bigcup\{N(\alpha, f(\alpha)) : \alpha \in A\} \cap \bigcup\{N(\beta, f(\beta)) : \beta \in \omega_1 - A\} = 0.$$

Then there is a $g : \omega_1 \rightarrow \omega$ and a closed unbounded $C \subseteq \omega_1$, such that:

$$\text{whenever } \alpha < \beta \text{ and } C \cap (\alpha, \beta] \neq \emptyset, \text{ then } N(\alpha, g(\alpha)) \cap N(\beta, g(\beta)) = \emptyset.$$

It should be clear that Lemma 14 yields \aleph_1 -collectionwise Hausdorffness in first countable normal spaces: without loss of generality we may assume the topology is on an ordinal and hence on a member of V ; let the $N(\alpha, i)$'s be a descending neighborhood base at α , where we have labeled the points of a discrete closed subspace of the space X with the countable ordinals. Define $c : \omega_1 \rightarrow \omega_1$ by letting $c(\alpha) = \sup(C \cap \alpha)$, and let $\alpha \sim \beta$ if $c(\alpha) = c(\beta)$. The \sim -classes are countable and normality implies \aleph_0 -collectionwise Hausdorffness, so there is a $q : \omega_1 \rightarrow \omega$ such that $c(\alpha) = c(\beta)$ implies $N(\alpha, q(\alpha)) \cap N(\beta, q(\beta)) = 0$. Let $r(\alpha) = \max(g(\alpha), q(\alpha))$. Then $\{N(\alpha, r(\alpha))\}_{\alpha < \omega_1}$ is the required separation.

Proof of Lemma 14. Let S be a Suslin tree. Let $\{\dot{N}(\alpha, i) : i < \omega, \alpha < \omega_1, \alpha < \omega_1\}$ be S -names for subsets of X as in the hypothesis. For $s \in S$, let $\ell(s)$ be the length of s . Since S has countable levels and its corresponding forcing poset is ω -distributive, we can construct an increasing function $h : \omega_1 \rightarrow \omega_1$ such that:

For all $\alpha < \omega_1$ and all $s \in S$ with $\ell(s) = h(\alpha)$, s decides all statements of the form “ $\dot{N}(\beta, j) \cap \dot{N}(\alpha, i) = 0$ ”, for all $i, j < \omega$ and $\beta < \alpha$.

Let \dot{A} be an S -name for a subset of ω_1 such that for no $\alpha < \omega_1$ does any $s \in S$ with $\ell(s) = h(\alpha)$ decide whether $\alpha \in \dot{A}$. To define such an \dot{A} , for each $\alpha < \omega_1$ pick two successors of each $s \in S$ with $\ell(s) = h(\alpha)$ and let one force $\alpha \in \dot{A}$ and let the other force $\alpha \notin \dot{A}$.

Let \dot{f} be an S -name for a function $f : \omega_1 \rightarrow \omega_1$ as in the hypothesis of the lemma. Let C be a closed unbounded subset of ω_1 in V such that for each $s \in S$ with $\ell(s) \in C$, s decides $f|_{\ell(s)}$ and $A|_{\ell(s)}$, and such that for all $\alpha < \beta < \omega_1$, if $\beta \in C$ then $h(\alpha) < \beta$. We will define an S -name \dot{g} for a function from ω_1 to ω such that whenever $\alpha < \beta < \omega_1$,

$$\text{if } (\alpha, \beta] \cap C \neq \emptyset, \text{ then } N(\alpha, g(\alpha)) \cap N(\beta, g(\beta)) = 0.$$

Let $c : \omega_1 \rightarrow \omega_1$ be defined by $c(\alpha) = \sup(C \cap \alpha)$. Fix $\beta < \omega_1$. Each $s \in S$ with $\ell(s) = h(\beta)$ decides $f|_{c(\beta)}$ and $A|_{c(\beta)}$ and “ $\dot{N}(\alpha, f(\alpha)) \cap \dot{N}(\beta, i) = 0$ ” for all $i < \omega$,

$\alpha < c(\beta)$, but not whether $\beta \in A$. Fix $s \in S$ with $\ell(s) = c(\beta)$. Since s does not decide whether $\beta \in A$, we claim that there is an $i_0 < \omega$ such that:

for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \in \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, i_0) = \emptyset$.

To see this, extend s to $t \in S$ forcing that $\beta \notin A$ and deciding $f(\beta)$. Let i_0 be the value of $f(\beta)$ as decided by t . Then t forces that $N(\alpha, f(\alpha)) \cap N(\beta, i_0) = \emptyset$ for all $\alpha < c(\beta)$, but these facts were already decided by s . Similarly, there is an $i_1 < \omega$ such that:

for all $\alpha < c(\beta)$ such that $s \Vdash \alpha \notin \dot{A}$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, i_1) = \emptyset$.

Since s decides $A|c(\beta)$, letting $\bar{i} = \max\{i_0, i_1\}$,

for all $\alpha < c(\beta)$, $s \Vdash \dot{N}(\alpha, \dot{f}(\alpha)) \cap \dot{N}(\beta, \bar{i}) = \emptyset$.

We have such an \bar{i}_s for each s in the $c(\beta)$ -th level of the tree, so we can construct a name \dot{g} such that:

$$s \Vdash \dot{g}(\beta) = \max\{\bar{i}_s, \dot{f}(\beta)\}$$

for each $s \in S$ with $\ell(s) = c(\beta)$. Then \dot{g} is as required. \square

We are now ready for topological applications. First, let us note that the procedures applied above in a PFA(S) context also work for $\text{MA}_{\omega_1}(\text{S})$. That is, do the Easton forcing, the $\text{MA}_{\omega_1}(\text{S})$ forcing, and the S -forcing and one obtains a model in which normal first countable spaces are collectionwise Hausdorff. By Lemma 8, we are then in position to establish Theorem 2. To prove it, let X be a hereditarily normal first countable countable chain condition space. Let $Y \subseteq X$. Since X is hereditarily collectionwise Hausdorff, Y cannot have an uncountable discrete subspace. It is standard that Y must therefore have a dense hereditarily Lindelöf subspace. For recursively define x_α such that $x_\alpha \notin \overline{\{x_\beta : \beta < \alpha\}}$, until for some λ , $\{x_\alpha : \alpha < \lambda\}$ is dense. Claim $\{x_\alpha : \alpha < \lambda\}$ is hereditarily Lindelöf. For if not, there would exist $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$ such that $\{x_{\alpha_\gamma} : \gamma < \delta\}$ is open in $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$, for each $\delta < \omega_1$. But then $\{x_{\alpha_\gamma} : \gamma < \omega_1\}$ is discrete.

Thus, since $\{x_\alpha : \alpha < \lambda\}$ is dense and hereditarily Lindelöf, by Lemma 8 Y is separable.

In the $\text{MA}_{\omega_1}(\text{S})$ situation, one can actually rely on L rather than on Easton and so the argument can be simplified somewhat.

The compatibility of “locally countable subspaces of size \aleph_1 in a compact countably tight space are σ -discrete” with “normal first countable spaces are collectionwise Hausdorff” enables us to strengthen a variety of results of Balogh [B] and other authors, in particular proving Theorem 1, which we shall now establish.

Following Nyikos [N₂], we have:

Definition. A space X is of **Type I** if $X = \bigcup_{\alpha < \omega_1} U_\alpha$, where the U_α 's are open, $\overline{U}_\beta \subseteq U_\alpha$ whenever $\beta < \alpha$, $U_\alpha = \bigcup_{\beta < \alpha} U_\beta$ for limit α , and each \overline{U}_α is Lindelöf. The **skeleton** of (such a decomposition of) a Type I space is the sequence $\{\overline{U}_\alpha - U_\alpha : \alpha < \omega_1\}$. The $\overline{U}_\alpha - U_\alpha$'s are called **bones**. We extend Nyikos' metaphor by calling a selection of one point from each bone a **bone-scan**.

We will be able to get away with referring to “the” skeleton since

Lemma 15 [N₂]. *Any two skeletons of a Type I space agree on a closed unbounded set of bones.*

The relevance of Type I spaces to the problem at hand is that:

Lemma 16. *If X is hereditarily collectionwise Hausdorff, locally hereditarily Lindelöf and subspaces of X are hereditarily Lindelöf if and only if they are hereditarily separable, then X is the disjoint union of clopen Type I spaces.*

To see that this result applies in our setting, let X be locally compact and perfectly normal. Then X is first countable and hereditarily normal, so it will be hereditarily collectionwise Hausdorff in our model. X is also locally hereditarily Lindelöf. Let $Y \subseteq X$. Whether Y is hereditarily Lindelöf or hereditarily separable, it has no uncountable discrete subspace. Since \overline{Y} is collectionwise Hausdorff, it also has no uncountable discrete subspace, and the same holds for its one-point compactification. By a well-known result of Arhangel'skiĭ, (see e.g. [H]) it follows that that compactification has countable tightness. But then by Theorem 7, Y is hereditarily separable if and only if it is hereditarily Lindelöf.

Proof of Lemma 16. X has a basis of open sets which are hereditarily Lindelöf and hereditarily separable. We first claim that *hereditarily Lindelöf open sets have hereditarily Lindelöf closure*. To see this, let U be hereditarily Lindelöf and hence hereditarily separable. It follows as usual, by collectionwise Hausdorffness, that \overline{U} has no uncountable discrete subspace. It follows that to show \overline{U} is hereditarily separable and hence hereditarily Lindelöf, it suffices to show each subspace Z of it is locally separable. Without loss of generality, let $W \cap Z$ be a neighborhood of z in Z , where W is a basic open neighborhood of Z in X . But then W is hereditarily separable, so $W \cap Z$ is separable.

Toward completing the proof of Lemma 18, let \mathcal{U}_0 be a maximal disjoint collection of basic open sets in X . Suppose \mathcal{U}_β , $\beta < \alpha$ have been defined to be unions of countably many disjoint collections of basic open sets. Let $F_\alpha = X - \bigcup_{\beta < \alpha} \bigcup \mathcal{U}_\beta$.

Let \mathcal{V}_α be a collection of basic open sets such that $\{F_\alpha \cap V : V \in \mathcal{V}_\alpha\}$ is a maximal disjoint collection of relatively open subsets of F_α . Fix a dense countable subset D_V in each $F_\alpha \cap V$. Any selection of points, one from each D_V , yields a discrete subset of $\bigcup \mathcal{V}_\alpha$, which may therefore be separated by basic open sets. We may

therefore cover $\bigcup\{D_V : V \in \mathcal{V}_\alpha\}$ by a collection \mathcal{U}_α of basic open sets, such that \mathcal{U}_α is the union of countably many collections of disjoint basic open sets. Claim $X = \bigcup_{\alpha < \omega_1} \bigcup \mathcal{U}_\alpha$. Suppose $x \notin \bigcup_{\alpha < \omega_1} \bigcup \mathcal{U}_\alpha$. Let W be a basic open neighborhood of x . Since the F_α 's are descending and W is hereditarily Lindelöf, there is an $\alpha < \omega_1$ such that $F_\alpha \cap W = F_{\alpha+1} \cap W$. Then $\bigcup \mathcal{U}_{\alpha+1} \cap W$ is empty, so $F_{\alpha+1}$ is dense in F_α . But that's a contradiction, since $x \in \bigcap_{\alpha < \omega_1} F_\alpha$.

$\bigcup_{\alpha < \omega_1} \mathcal{U}_\alpha$ is the union of \aleph_1 collections of disjoint open sets. Each member of $\bigcup_{\alpha < \omega_1} \mathcal{U}_\alpha$ meets only countably many elements of each such collection. Therefore X is the sum of clopen subspaces, each composed of the union of \aleph_1 basic open sets. Let one of those clopen subspaces, say S , be such that $S = \bigcup_{\alpha < \omega_1} S_\alpha$, S_α basic open.

We work within S . Let $T_0 = S_0$. Suppose $\{T_\beta\}_{\beta < \alpha}$ have been defined so that each T_β is open with hereditarily Lindelöf closure. For α limit, let $T_\alpha = \bigcup_{\beta < \alpha} S_\beta \cup \bigcup_{\beta < \alpha} T_\beta$.

Then $\overline{T_\alpha}$ is hereditarily Lindelöf. For $\alpha = \beta + 1$, since $S_\beta \cup T_\beta$ is hereditarily Lindelöf and hence has (hereditarily) Lindelöf closure, pick basic open $\{W_\alpha^n\}_{n < \omega}$ such that $\overline{S_\beta \cup T_\beta} \subseteq \bigcup_{n < \omega} W_\alpha^n$. Let $T_\alpha = \bigcup_{n < \omega} W_\alpha^n$. Since for any α , we have $\overline{T_\alpha} \subseteq T_{\alpha+1}$, and $\bigcup_{\alpha < \omega_1} T_\alpha = S$, we have shown S is Type I.

In [N₂] Nyikos erroneously ascribes to Gruenhage [G₂] the assertion that *every locally compact perfectly normal collectionwise Hausdorff space is the disjoint union of clopen type I spaces*. Ostaszewski's space [0] is in fact a counterexample. What Gruenhage in fact proved was that *every locally compact perfectly normal space which is collectionwise normal with respect to compact sets is the disjoint union of clopen subspaces, each of which is the union of \aleph_1 open subspaces, each with compact closure*. The hypothesis of collectionwise normality with respect to compact sets was later weakened by Junnila (unpublished) to collectionwise Hausdorffness. Under MA_{ω_1} – which was the situation of interest in [N₂] – indeed the ascribed assertion holds by Lemma 16, since perfectly normal collectionwise Hausdorff spaces are hereditarily collectionwise Hausdorff, and the equivalence between hereditary Lindelöfness and hereditary separability follows from MA_{ω_1} the same way we did it here. However we think the finer analysis of Lemma 18 is interesting. The proof technique can be found in [B].

Since the disjoint union of clopen paracompact spaces is paracompact, given Lemma 16, we may confine ourselves to considering Type I spaces. For the same reason, we note:

Lemma 17 [N₂]. *If the skeleton of a Type I space has a closed unbounded set of empty bones, the space is paracompact.*

Nyikos further notes:

Lemma 18 [N₂]. *If X is locally hereditarily Lindelöf Type I, then X is hereditarily collectionwise Hausdorff if and only if every discrete subspace misses the elements of a skeleton closed unboundedly often.*

This is proved by a standard pressing-down argument. Thus, to prove Theorem 1, it suffices to observe that bone-scans are locally countable and that Theorem 6 applies, for then some bone-scan will meet stationarily many bones unless all bone-scans are countable, in which case the space is Lindelöf and hence paracompact. To see that Theorem 6 does apply, first recall

Definition. $f : X \rightarrow Y$ is **perfect** if it is continuous, closed, and inverse images of points are compact.

The same argument that proves that the set of limit ordinals in ω_1 is not a G_δ extends to show that:

Lemma 19 [B]. *A perfectly normal space does not include a perfect pre-image of ω_1 .*

Combining this with another lemma of Balogh will then finish the proof of Theorem 1:

Lemma 20 [B]. *If X is locally compact and countably tight, then the one-point compactification of X is countably tight if and only if X does not include a perfect preimage of ω_1 .*

THEOREM 2

There are a number of consequences of the compatibility of “locally countable subspaces of countably tight compact spaces are σ -discrete” with “all normal first countable spaces are collectionwise Hausdorff” that follow relatively straightforwardly from Balogh’s work [B], for example:

Theorem 21. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact perfectly normal space of cardinality \aleph_1 is metrizable.*

Proof. We first note a space such as in the Theorem has a countable neighborhood around each point. This is because PFA(S) implies $2^{\aleph_0} = \aleph_2$, S -forcing satisfies the countable chain condition, and compact first countable spaces have cardinality either \aleph_0 or 2^{\aleph_0} . But countable compact sets are metrizable and paracompact locally metrizable spaces are metrizable, so we are done.

Balogh [B] proved under MA + \sim CH that connected, locally compact, locally hereditarily Lindelöf, hereditarily normal collectionwise Hausdorff spaces are paracompact if and only if they do not include a perfect pre-image of ω_1 . We drop two of these conditions and get:

Theorem 22. *If it is consistent there is a supercompact cardinal, it is consistent that locally compact, locally hereditarily Lindelöf, hereditarily normal spaces are paracompact if and only if they do not include a perfect pre-image of ω_1 .*

This answers a question Balogh asked for manifolds in [B]. Theorem 22 will follow immediately from the following lemma, which Balogh [B] proved from MA_{ω_1} , but just using the consequences mentioned.

Lemma 23. *Suppose first countable hereditarily Lindelöf spaces are hereditarily separable and locally countable subspaces of size \aleph_1 of a compact countably tight space are σ -discrete. Then if X is locally hereditarily Lindelöf, hereditarily collectionwise Hausdorff, and can be embedded into a countably tight compact space, then X is paracompact.*

This is proved by the same argument as for Theorem 1. In the situation of Theorem 22, we know the one-point compactification of X is countably tight, that X is first countable, and hence that X is hereditarily collectionwise Hausdorff. So Lemma 23 applies.

As a corollary, we will get a metrization theorem which answers a question in [BB]:

Definition. X has a \mathbf{G}_δ -diagonal if $\{\langle x, x \rangle : x = x\}$ is a G_δ in $X \times X$.

Lemma 24 [C]. *A countably compact space with a G_δ -diagonal is metrizable (and hence compact.)*

Theorem 25. *If it is consistent there is a supercompact cardinal, it is consistent that every locally compact hereditarily normal space with a G_δ -diagonal is metrizable.*

Proof. It suffices to show X is paracompact, since it is locally metrizable. It follows from Lemma 24 that X does not include a perfect pre-image of ω_1 , as well as that X is first countable and locally hereditarily Lindelöf. But then Lemma 23 applies.

In [LT2], Larson and Todorćević used S -forcing over $\text{MA}_{\omega_1}(S)$ in order to prove the consistency of *every compact space with hereditarily normal square is metrizable*. We shall extend this to locally compact spaces by using our model, obtaining

Theorem 26. *If it's consistent there is a supercompact, it's consistent that every locally compact space with hereditarily normal square is metrizable.*

As usual, we don't know if the supercompact is necessary – we doubt it – but we think it unlikely that the locally compact result can be obtained from the compact case for the following reason. Katětov [Ka] proved that every compact space with hereditarily normal cube is metrizable. There is no such ZFC result for locally compact spaces – it is routine to show:

Theorem 27. MA_{ω_1} implies there is a locally compact non-metrizable space X with X^n hereditarily normal for all $n \in \omega$.

Proof. This is standard. X will be any subset of the real line of size \aleph_1 with the following topology. Let D be countable dense in X in the real line subspace topology. Each point of D we make isolated. For each $x \in X - D$, we fix a sequence from D converging to x , and let a neighborhood of x be $\{x\}$ together with a tail of the sequence. Then X is locally compact and non-metrizable, as is X^n , for each $n \in \omega$. X^n has a weaker separable metrizable topology, as a subspace of \mathbb{R}^n . Each point of X^n has a neighborhood base consisting of sets which are compact in that weaker topology. By the following lemma, MA_{ω_1} will imply X^n is hereditarily normal.

Lemma 28 [We, 7.1]. Assume $MA(\sigma\text{-centred}) + \sim CH$. Suppose ρ and τ are two topologies on a set X such that

- (i) $\rho \subseteq \tau$.
- (ii) $\langle X, \rho \rangle$ is Hausdorff and second countable, and
- (iii) there is a closed neighborhood base for τ consisting of sets compact in $\langle X, \rho \rangle$.

Then for all $H, K \in [X]^{<2^{\aleph_0}}$ such that $\overline{H} \cap K = H \cap \overline{K} = \emptyset$ in the τ topology, we have disjoint open U_H and U_K in τ including H and K respectively.

Now to prove Theorem 26, we will need a lemma of Katětov [Ka]:

Lemma 29. If Y is countably compact and Y^2 is hereditarily normal, then Y is perfectly normal.

Now suppose X is locally compact and X^2 is hereditarily normal. By [LT2] it follows that X is locally metrizable and hence first countable and locally hereditarily Lindelöf. X is homeomorphic to a subspace of X^2 , so it too is hereditarily normal. Therefore it is hereditarily collectionwise Hausdorff. Since X is locally metrizable, to show it is metrizable it suffices to show it is paracompact. This will follow from Lemma 20 if we can show X includes no perfect pre-image of ω_1 . Suppose it had such a pre-image Y . Y would be countably compact and, by Katětov's Lemma, perfectly normal. But that's impossible by Lemma 19.

Remarks. According to Todorćević (personal communication) the supercompact can probably be eliminated in the work of his on which Theorem 1 depends. We are much less optimistic about lowering it significantly in Theorem 4. See [N₃] for unsuccessful attempts to lower it in his proofs.

5. MAMA

It is interesting to note that the model for Theorem 2 cannot be used to get that first countable hereditarily normal countable chain condition spaces are hereditarily Lindelöf. This may well be consistent, but stronger methods are needed. We shall prove:

Theorem 30. *Assume $2^{\aleph_1} = \aleph_2$. There is a c.c.c. poset $Q * \dot{R}$ of size \aleph_2 such that after forcing with any c.c.c. poset P_0 , there is a first countable perfectly normal hereditarily separable space which is not Lindelöf.*

This does it, since one can start e.g. with L and in the $Q * \dot{R}$ extension let $P_0 = P_1 \times P_2 \times S$, where P_1 is c.c.c. and forces a (coherent) Suslin tree and P_2 forces $\text{MA}_{\omega_1}(S)$. In order to force $\text{MA}_{\omega_1}(S)$ we need $2^{\aleph_1} \leq \aleph_2$, but $Q * \dot{R} * \dot{P}_1$ preserves this.

Proof. We actually just need to assemble results of others.

Lemma 31 [Lar, LarTo]. *Over $\text{MA}_{\omega_1}(S)$, S forces $\mathfrak{b} > \aleph_1$.*

Lemma 32 [vD]. *$\mathfrak{b} > \aleph_1$ implies that in a first countable regular space of size \aleph_1 , disjoint closed sets, one of which is countable, have disjoint open sets about them.*

Lemma 33 [So]. *$2^{\aleph_1} = \aleph_2$ implies there is a c.c.c. poset $Q * \dot{R}$ of size \aleph_2 such that after forcing with any c.c.c. poset P_0 , there is a first countable 0-dimensional space of size \aleph_1 in which every open set is countable or co-countable.*

It just remains to show Soukup's space has the desired properties in our model. He notes it is hereditarily separable but not Lindelöf; by Lemma 32, it is hereditarily normal. Without loss of generality, by passing to a subspace if necessary, we may assume the space is locally countable. But as Roitman [R, p. 314] notes, *a countable subset of a locally countable space is a G_δ* . Since cocountable sets are also G_δ 's, we see that the space is perfectly normal.

Of course the natural candidate for a model in which first countable hereditarily normal countable chain condition spaces are both hereditarily Lindelöf and hereditarily separable is the one in which $\text{PFA}(S)$ is used instead of $\text{MA}_{\omega_1}(S)$. We do not know the answer to the following:

Problem. *Force with S over $\text{PFA}(S)$. Are there any first countable regular hereditarily separable non-Lindelöf spaces in the resulting model?*

We now come to the second half of the paper. Nyikos [N₁] and [N₂] raised the question of whether it is consistent that every hereditarily normal manifold of dimension > 1 is metrizable. He later showed that the answer is positive under strong set-theoretic axioms consistent from a supercompact cardinal, provided the space is also hereditarily collectionwise Hausdorff [N₃], [N₄]. It is natural to wonder whether our methods can solve this question as well, and the answer is that we can drop the additional topological condition to get:

Theorem 34. *It is consistent there is a supercompact cardinal, it is consistent that every hereditarily normal manifold of dimension > 1 is metrizable.*

Our official definition of “manifold” here is a connected space in which each point has a neighborhood homeomorphic to some E^n . In particular then, manifolds are locally compact metrizable, so are first countable and locally hereditarily Lindelöf.

We are able to drop Nyikos’ “hereditarily collectionwise Hausdorff” because manifolds are first countable. We will work in a model that has all the features of the one for Theorem 1, plus more. In [N₃] Nyikos first proves:

Nyikos’ Lemma 2.5. *MA_{ω_1} implies every hereditarily collectionwise Hausdorff manifold is of Type I.*

We have the conclusion in our situation by the argument following the statement of Lemma 18, and the fact that manifolds are connected.

Nyikos then employs two axioms concerning ideals: “CC₂₂” and “SSA”, which he notes are consistent with PFA.

Definition. $\mathcal{I} \subseteq \mathcal{P}(X)$ is an **ideal** if every subset of a member of \mathcal{I} is in \mathcal{I} , and \mathcal{I} is closed under finite unions. An ideal \mathcal{I} of countable subsets of X is **countable-covering** if for each countable $Q \subseteq X$, there are $\{I_n^Q : n \in \omega\}$, each $I_n^Q \in \mathcal{I}$, such that whenever $I \subseteq Q$ and $I \in \mathcal{I}$, then $I \subseteq I_n^Q$ for some n . **CC₂₂** is the assertion that for each countable-covering ideal \mathcal{I} on a stationary subset S of ω_1 , either

- (i) there is a stationary $A \subseteq S$ such that $[A]^\omega \subseteq \mathcal{I}$,
- or (ii) there is a stationary $B \subseteq S$ such that $B \cap I$ is finite for all $I \in \mathcal{I}$.

Definition. Let $S \subseteq \omega_1$ be stationary. The ideal of nonstationary subsets of S is $(\aleph_2, \aleph_2, \aleph_0)$ -**saturated** if for every collection \mathcal{Z} of \aleph_2 stationary subsets of S , there is a $\mathcal{W} \subseteq \mathcal{Z}$ of size \aleph_2 , such that every countable subcollection of \mathcal{W} has stationary intersection. The ideal is **weakly** (\aleph_2, \aleph_0) -**saturated** if for every collection \mathcal{Z} of \aleph_2 stationary subsets of S , there is a countable $\mathcal{Z}' \subseteq \mathcal{Z}$ such that $\bigcap \mathcal{Z}'$ is uncountable. **SSA (PSA)** is the assertion that there is a stationary $S \subseteq \omega_1$ such that the ideal of nonstationary subsets of S is $(\aleph_2, \aleph_2, \aleph_0)$ -saturated (weakly (\aleph_2, \aleph_0) -saturated).

Nyikos does not really need SSA in his proof – PSA suffices. We don’t know if we can get SSA to hold in the following result, but we don’t need to.

We will show:

Theorem 35. *It is consistent with $PFA(S)$ that S forces CC_{22} , PSA , and “normal first countable spaces are collectionwise Hausdorff”.*

In his proof of metrizability, Nyikos needs a fact specific to manifolds of dimension > 1 :

Nyikos' Theorem 2.6. *If M is a Type I manifold of dimension > 1 , then a skeleton may be chosen so that each point of a bone is contained in a compact connected infinite subset of the bone.*

Nyikos uses his 2.6 and a result of Balogh []:

PFA implies every first countable, countably compact non-compact space includes a copy of ω_1 ,

in order to obtain:

Nyikos' Theorem 2.7. *$CC_{22} + PFA$ imply that if M is a normal hereditarily collectionwise Hausdorff manifold with dimension > 1 , then M is either metrizable or includes a family of \aleph_2 disjoint copies of ω_1 .*

He then notes that by additionally assuming SSA, the proof of his 2.7 can be converted into a proof that:

Nyikos' Theorem. *$CC_{22} + PFA + SSA$ imply that if M is a hereditarily normal, hereditarily collectionwise Hausdorff manifold with dimension > 1 , then X is metrizable.*

The conjunction of these hypotheses is consistent, assuming the consistency of a supercompact $[N_3]$, $[EN]$. There are several difficulties in trying to adapt Nyikos' proof to our situation in which we have hereditary collectionwise Hausdorffness for free; for one thing, we don't know whether Balogh's result is S -forced over $PFA(S)$. However, we can replace the copies of ω_1 in Nyikos' proof with perfect pre-images of ω_1 . The advantage of dealing with perfect pre-images of ω_1 is the following ZFC result due to Gruenhage $[G_1]$ as well as to D. Burke (unpublished).

Definition. *A space is ω -bounded if every countable subset has compact closure.*

Lemma 36. *If X is ω -bounded, it is either compact or includes a perfect pre-image of ω_1 .*

It is also useful to note:

Lemma 37. *A perfect pre-image of ω_1 is countably compact but not compact. Furthermore, in a first countable space, such a pre-image – indeed any countably compact subspace – is closed.*

The proof is left to the reader.

We will now prove Theorem 34, assuming we can vary the model of the first part of the paper so as to have CC_{22} and PSA hold in addition.

By Nyikos' Lemma 2.5, we may assume our manifold M is of type I, say $M = \bigcup \{M_\alpha : \alpha < \omega_1\}$. By his 2.6, we may choose a skeleton so that each point of a

bone $B_\alpha = \overline{M}_\alpha - M_\alpha$ is contained in a compact connected infinite subset of the bone. Since manifolds are connected, each bone is non-empty. Let $\{x_\beta\}_{\beta < \omega_1}$ be a bone-scan.

For any $T \subseteq \omega_1$ let $\hat{T} = \{x_\beta : \beta \in T\}$. Let S be a stationary subset of ω_1 and let \mathcal{I} be the ideal of all countable subsets I of S such that \hat{I} has compact closure. We claim \mathcal{I} is countable-covering. The point is that every countable subset P of M has Lindelöf closure and so \overline{P} can be covered by an increasing chain of countably many open sets $\{V_n\}_{n < \omega}$, each with compact closure. Then any subset of P with compact closure is a subset of some V_n . It follows that if Q is a countable subset of S , and we define J_n^Q so that $\hat{J}_n^Q = V_n \cap \hat{Q}$, then every $I \in \mathcal{I}$ which is a subset of Q is included in J_n^Q for some n . Notice for future reference that if M' is a subspace of M , $\hat{S} \subseteq M'$, and \mathcal{I}' is the ideal of all countable subsets I of S such that \hat{I} has compact closure in M' , then the same argument shows \mathcal{I}' is countable-covering. Now apply CC_{22} to the ideal \mathcal{I} of countable subsets I of the stationary set S of PSA such that \hat{I} is compact. Then if $A \subseteq S$ is such that $A \cap I$ is finite for all $I \in \mathcal{I}$, then \hat{A} is closed discrete. For if not, by first countability, there would be a non-trivial convergent sequence $\{x_{\alpha_n}\}_{n < \omega}$, $\alpha_n \in A$. But then $\overline{\{x_{\alpha_n} : n < \omega\}}$ is compact and so A has infinite intersection with $I = \{\alpha_n : n < \omega\} \in \mathcal{I}$. Such an A then cannot be stationary, else hereditary collectionwise Hausdorffness and the Pressing Down Lemma would yield uncountably many disjoint open subsets of some M_α . But the M_α 's are separable. It follows that there is a stationary $S' \subseteq S$ such that each countable subset of \hat{S}' has compact closure. Claim $\overline{\hat{S}'}$ is ω -bounded. Let Z be a countable subset of \hat{S}' . Claim \overline{Z} – which is just the closure of Z in \hat{S}' – is compact. By first countability, \overline{Z} is included in some $\overline{Z'}$, where Z' is a countable subset of \hat{S}' . But then \overline{Z} is compact.

Now $\hat{S}' \subseteq \bigcup_{\alpha < \omega_1} B_\alpha$; by first countability $\overline{\hat{S}'} = \bigcup \{\overline{E} : E \in [\bigcup_{\alpha < \omega_1} \hat{S}']^{\aleph_0}\}$. Each such E is either included mod finite in some B_α , or not. In either case, each element of $\overline{E} - E$ is an element of some B_β , so $\hat{S}' \subseteq \bigcup_{\alpha < \omega_1} B_\alpha$. It follows, since \hat{S}'

meets stationarily many B_α , that $\overline{\hat{S}'}$ is not compact, because \hat{S}' has no complete accumulation point in $\overline{\hat{S}'}$. By Lemma 32, $\overline{\hat{S}'}$ therefore includes a perfect pre-image W of ω_1 . W is then countably compact, closed, but not compact. It follows that $S''' = \{\alpha : W \cap B_\alpha \neq \emptyset\}$ – which is included in S' – is stationary, else W would be paracompact. Also note that $M - W$ is open and so is locally compact. For each α such that $W \cap B_\alpha \neq \emptyset$, pick a compact infinite connected $K_\alpha \subseteq B_\alpha$ containing some point w_α in $W \cap B_\alpha$. Pick $y_\alpha \in K_\alpha$, $y_\alpha \neq w_\alpha$. By the usual C_{22} argument, we can find a stationary $S'''' \subseteq S'''$ such that the closure of $\hat{S}'''' = \{y_\alpha : \alpha \in S''''\}$ in $M - W$ is ω -bounded, and hence a perfect pre-image Y of ω_1 , $Y \subseteq$ the closure of \hat{S}'''' in $M - W$. By Lemma 33, Y is closed in M . It is disjoint from W , so by Urysohn's Lemma, there is a continuous $f : M \rightarrow [0, 1]$ such that f sends W to 0 and Y to 1. Since M is connected, f is onto. Since each K_α is connected, for each

$r \in [0, 1]$, $f^{-1}\{r\} \cap K_\alpha \neq 0$, else $f^{-1}([0, r))$ and $f^{-1}((r, 1])$ would disconnect K_α . The same argument as before then produces for each $r \in [0, 1]$ a stationary $S_r \subseteq S'''$ and a perfect pre-image F_r of ω_1 , $F_r \subseteq f^{-1}\{r\}$, F_r meeting all K_α , $\alpha \in S_r$. (We won't need it, but note we have established a version of Nyikos' 2.7, in which we have substituted "perfect pre-images" for "copies" of ω_1 .) Pick $p_\alpha^r \in F_r \cap K_\alpha$ for each $r \in [0, 1]$ and $\alpha \in S_r$.

We now apply *PSA* to $\{S_r : r \in [0, 1]\}$, noting that, by Lemma 12, $2^{\aleph_0} = \aleph_2$. Let $\{r_n : n \in \omega\}$ be any infinite subset of $[0, 1]$ such that $A = \bigcap \{S_{r_n} : n \in \omega\}$ is uncountable. Let $r \in \overline{\{r_n : n \in \omega\}}$. We claim:

$$M^* = M - (f^{-1}\{r\} \cap \bigcup \{B_\alpha : \alpha \in A'\}),$$

(where A' is the derived set of A) is not normal, contradicting M hereditarily normal. Specifically, we claim the following two closed subsets of M^* witness non-normality:

$$H = M^* \cap \bigcup \{B_\alpha : \alpha \in A'\},$$

and

$$K = M^* \cap f^{-1}\{r\}.$$

Suppose U is an open subset of M^* including K . Claim $\overline{U} \cap H \neq 0$. Because K meets each K_α such that $\alpha \in A - A'$, it follows that for each $\alpha \in A - A'$, there exists $n(\alpha) \in \omega$ such that

$$f^{-1}\left(r - \frac{1}{n(\alpha)}, r + \frac{1}{n(\alpha)}\right) \cap K_\alpha \subseteq U.$$

For if not, we could pick $z_n \in K_\alpha \cap f^{-1}\left(r - \frac{1}{n}, r + \frac{1}{n}\right)$ outside of U and these would accumulate at a point of K , contradicting U open.

Since $A - A'$ is uncountable, there is a k such that $n(\alpha) = k$ for uncountably many $\alpha \in A - A'$. Among those α , pick an ascending sequence of α_i which converges to some δ . Then $\delta \in A'$ and $B_\delta - f^{-1}\{r\} \subseteq H$. Taking r_m such that $0 < |r - r_m| < \frac{1}{k}$, consider $\{p_{\alpha_i}^{r_m} : i < \omega\}$. Its closure is compact, but all its limit points are in the closed set $B_\delta \cap f^{-1}\{r_m\}$ and hence in H . Thus points of H are in \overline{U} , which was to be proved. Thus M^* cannot be normal.

It remains to verify that *PSA* and *CC*₂₂ hold in our model.

20. P₂₂ AND PFA⁺⁺(S)

Definition. An ideal $\mathcal{I} \subset [\omega_1]^{\aleph_0}$ is a *P-ideal* if for all countable $\mathcal{A} \subset \mathcal{I}$ there is a $B \in \mathcal{I}$ such that $A \subset^* B$ for all $A \in \mathcal{A}$.

Here $A \subset^* B$ means that the set $A \setminus B$ is finite.

Definition. The axiom P_{22} states that if $\mathcal{I} \subset [\omega_1]^{\aleph_0}$ is a P -ideal and $S \subset \omega_1$ is stationary then there is a stationary $S' \subset S$ such that either

- a) $[S']^{\aleph_0} \subset \mathcal{I}$, or
- b) for all $x \in [S']^{\aleph_0}$ and all $y \in \mathcal{I}$, $x \cap y$ is finite.

Given such an \mathcal{I} and S , we will call an S' as given by P_{22} a $P_{22}(\mathcal{I}, S)$ -witness.

Definition. The axiom $PFA^{++}(S)$ states that there is a coherent Suslin tree S and that for all proper partial orders P preserving S , if D_α ($\alpha < \omega_1$) are dense subsets of P and \dot{R}_α ($\alpha < \omega_1$) are P -names for stationary subsets of ω_1 , then there is a filter $G \subset P$ such that for all $\alpha < \omega_1$, $G \cap D_\alpha \neq \emptyset$ and $\{\beta < \omega_1 \mid \exists p \in G \ p \Vdash \beta \in \dot{R}_\alpha\}$ is stationary.

Theorem 41. (CH) *If S is a coherent Suslin tree and $\dot{\mathcal{I}}$ is an S -name for a P -ideal and \dot{R} is an S -name for a stationary subset of ω_1 , then there is an S -preserving proper forcing adding an S -name for a $P_{22}(\mathcal{I}, R)$ -witness.*

One key point is that under CH any P -ideal $\mathcal{I} \subset [\omega_1]^{\aleph_0}$ is generated by a sequence $\{A_\alpha : \alpha < \omega_1\} \subset \mathcal{I}$ such that each $A_\alpha \subset \alpha$ and $\alpha < \beta \Rightarrow A_\alpha \subset^* A_\beta$.

Lemma 42. *If S is a coherent Suslin tree, $\dot{\mathcal{I}}$ is an S -name for a P -ideal and \dot{R} is an S -name for a stationary subset of ω_1 , then $S, \dot{\mathcal{I}}$ and \dot{R} are still these things after forcing with $Coll(\omega_1, \mathbb{R})$.*

This follows easily from the following facts. (1) The forcing $Coll(\omega_1, \mathbb{R})$ preserves stationary subsets of ω_1 and doesn't add countable sets of ordinals. (2) If S is an Aronszajn tree, an S -name \dot{R} for a stationary subset of ω_1 is witnessed by ω_1 -many stationary sets, the sets $\{\alpha \mid s \Vdash \check{\alpha} \notin \dot{R}\}$ for each $s \in S$.

Putting these together, we have the following.

Lemma 43. ($PFA^{++}(S)$) *If S is a coherent Suslin tree, $\dot{\mathcal{I}}$ is an S -name for a P -ideal and \dot{R} is an S -name for a stationary subset of ω_1 , then there is an S -name for a $P_{22}(\mathcal{I}, R)$ -witness.*

The rest of this section concerns the proof of Theorem 41. We essentially follow the proof of Eisworth and Nyikos for showing that P_{22} is consistent with CH.

Fix functions $a_\alpha : \alpha \rightarrow \omega$ ($\alpha < \omega_1$) generating our coherent Suslin tree S and S -names \dot{A}_α ($\alpha < \omega_1$) for sets $A_\alpha \subset \alpha$ ($\alpha < \omega_1$) generating the P -ideal \mathcal{I} and satisfying $\alpha < \beta < \omega_1 \Rightarrow A_\alpha \subset^* A_\beta$. We may assume that for all $s \in S$ such that s decides the value of each A_α ($\alpha < l(s)$) that s decides $A_{l(s)}$ as well. To see this, note that since S is Suslin there are only ω many possibilities for $A_{l(s)}$ compatible with s , and each of these must contain each A_α , $\alpha < l(s)$ as decided by s . Letting $A_{l(s)}$ be any subset of $l(s)$ almost containing each of these A_α and almost contained in each of the original possibilities for $A_{l(s)}$ does not affect the ideal \mathcal{I} , then, as long as we shift the names to include the old $A_{l(s)}$ at a later stage.

A *promise* is an S -name \dot{f} for a function $f: E \rightarrow [\omega_1]^{<\omega}$ where E is a stationary subset of ω_1 and $f(\alpha) \subset \alpha$ for all $\alpha \in E$.

Our partial order P consists of conditions $p = (\alpha_p, \Phi_p, \{x_{p,s} : s \in S(\alpha_p)\})$ such that $\alpha_p \in \omega_1$, Φ_p is a countable set of promises and $x_{p,s}$ is a function from α_p to 2 such that s forces that $x_{p,s}^{-1}(\{1\})$ (we will also call this set $[x_{p,s}]$ and $[p, s]$) is a member of \mathcal{I} contained in R . Given conditions q and p , $q \leq p$ if $\alpha_q \geq \alpha_p$ and for all $s \in S(\alpha_q)$, $x_{q,s} \supset x_{p,s|_{\alpha_p}}$, $\Phi_q \supset \Phi_p$ and for all $\dot{f} \in \Phi_p$, s forces

$$\dot{Y}(\dot{f}, q, p, s) = \{\xi \in \text{dom}(\dot{f}) \mid [q, s] \setminus [p, s|_{\alpha_p}] \subset \dot{A}_\xi \setminus \dot{f}(\xi)\}$$

to be stationary.

We define the game $\Gamma(\delta, P', p)$, where $\delta < \omega_1$ is a limit ordinal, P' is a countable subset of P and p is a member of P' with $\alpha_p < \delta$, as follows:

Stage 0: $F_0 = \emptyset$, $p_0 = p$.

Stage $n + 1$: I chooses finite $F_{n+1} \supset F_n$ and II plays $p_{n+1} \leq p_n$ in P' such that $\alpha_{p_{n+1}} < \delta$ and for all $s \in S(\alpha_{p_{n+1}})$, if $s \setminus (s|_{\alpha_{p_n}}) = a_\delta|_{[\alpha_{p_n}, \delta)}$ then

$$s \Vdash [p_{n+1}, s] \setminus [p_n, s|_{\alpha_{p_n}}] \subset \dot{A}_\delta \setminus F_{n+1}.$$

I wins the play if $\{p_n : n \in \omega\}$ has a lower bound in P , or if $\sup\{\alpha_{p_n} : n \in \omega\} < \delta$.

Proposition 44. *If $\delta < \omega_1$ is a limit ordinal, P' is a countable subset of P and p is a condition in P' with $\alpha_p < \delta$, then I has a winning strategy in $\Gamma(\delta, P', p)$.*

Proof. Note that for all $s \in S(\delta)$,

$$s \Vdash \cup\{[p_n, s|_{\alpha_{p_n}}] : n \in \omega\} \subset \dot{A}_\delta \cup [p_{\bar{n}}, s|_{\alpha_{p_{\bar{n}}}}] \in \dot{\mathcal{I}},$$

where \bar{n} is the largest integer n such that $s|_{(\delta \setminus \alpha_{p_n})} \neq a_\delta|_{(\delta \setminus \alpha_{p_n})}$. So we need to see only that each such s forces that for each $n \in \omega$ and $\dot{f} \in \Phi_{p_n}$, the set

$$\dot{K}(\dot{f}, n, s) = \{\xi \in \text{dom}(\dot{f}) \mid [q, s] \setminus [p_n, s|_{\alpha_{p_n}}] \subset \dot{A}_\xi \setminus \dot{f}(\xi)\}$$

is stationary.

Note that for all $p'' \leq p'$ in P , all $s \in S(\alpha_{p''})$ and all $\dot{f} \in \Phi_{p'}$, $s \Vdash \dot{Y}(\dot{f}, p'', p', s)$ is stationary and $1_S \Vdash \dot{A}_\delta \subset^* A_\xi$ for all $\xi > \delta$. Therefore, s forces that there exists a finite $\dot{e}(\dot{f}, p'', p', s)$ such that

$$\dot{K}(\dot{f}, p'', p', s) = \{\xi \in \dot{Y}(\dot{f}, p'', p', s) \mid \dot{A}_\delta \setminus \dot{e}(\dot{f}, p'', p', s) \subset \dot{A}_\xi \setminus \dot{f}(\xi)\}$$

is stationary (we follow Eisworth and Nyikos here in using two predicates \dot{K} with different arities).

Let $\delta^* \geq \delta$ be such that every condition on level δ^* of S chooses a suitable value $\dot{e}(\dot{f}, p'', p', s)$ for $\dot{e}(\dot{f}, p'', p', s|_{\alpha_{p''}})$, for all $p'' \leq p'$ in P' , $s \in S(\alpha_{p'})$, $\dot{f} \in \Phi_{p'}$.

Now I divides his moves so that for each $s \in S(\delta^*)$, $n < \omega$ and $\dot{f} \in \Phi_{p_n}$ there are infinitely many $i \geq n$ such that I plays $F_{i+1} = F_i F_i \cup e(\dot{f}, p_i, p_n, s)$.

To see that this works, we show that 1_S forces that for each $\dot{f} \in \Phi_{p_n}$ there is an $i \geq n$ such that

$$(**) \dot{K}(\dot{f}, p_i, p_n, g(\alpha_{p_i})) \subset \dot{K}(\dot{f}, p_n, g(\delta)),$$

where $g(\alpha)$ denotes the member of the generic on the α th level. This will finish the proof.

To see that $(**)$ holds, let $s = g(\delta^*)$. Let i be the least integer $\geq n$ such that $s|[\alpha_{p_i}, \delta] = a_\delta|[\alpha_{p_i}, \delta]$ and $F_{i+1} = F_i \cup e(\dot{f}, p_i, p_n, s)$.

Now

$$[q_s|_{\alpha_q}] \setminus [p_n, s|_{\alpha_{p_n}}] = [q_s|_{\alpha_q}] \setminus [p_i, s|_{\alpha_{p_i}}] \cup [p_i, s|_{\alpha_{p_i}}] \setminus [p_n, s|_{\alpha_{p_n}}],$$

which s forces to be a subset of

$$\dot{A}_\delta \setminus F_{i+1} \cup [p_i, s|_{\alpha_{p_i}}] \setminus [p_n, s|_{\alpha_{p_n}}] \subset \dot{A}_\delta \setminus e(\dot{f}, p_i, p_n, s) \cup [p_i, s|_{\alpha_{p_i}}] \setminus [p_n, s|_{\alpha_{p_n}}].$$

If $\xi \in \dot{K}(\dot{f}, p_i, p_n, g(\alpha_{p_i}))$ then $\xi \in \dot{Y}(\dot{f}, p_i, p_n, g(\alpha_{p_i}))$, so

$$[p_i, s|_{\alpha_{p_i}}] \setminus [p_n, s|_{\alpha_{p_n}}] \subset \dot{A}_\xi \setminus \dot{f}(\xi).$$

Fixing such a ξ , s forces that

$$\dot{A}_\delta \setminus \dot{e}(\dot{f}, p_i, p_n, s|_{\alpha_{p_i}}) \subset \dot{A}_\xi \setminus \dot{f}(\xi),$$

so $[q, s] \setminus [p_n, s|_{\alpha_{p_n}}] \subset \dot{A}_\xi \setminus \dot{f}(\xi)$, so s forces that $\xi \in \dot{K}(\dot{f}, p_n, g(\delta))$. \square

Now, if $p \in P$ and \dot{H} is a P -name for a maximal antichain in S , and $D \subset P$ is dense open, then $\delta \in (\alpha_p, \omega_1)$ is *bad* for p, \dot{H} and D if there is a finite $F_\delta \subset \delta$ and a $\bar{s} \in S(\delta)$ such that for no $q \leq p$ with $q \in D$ is it true that for all $s \in S(\alpha)$ with $s|[\alpha_p, \alpha] = a_\alpha|[\alpha_p, \alpha]$

$$[q, s|_{\alpha_q}] \setminus [p, s|_{\alpha_p}] \subset \dot{A}_\alpha \setminus F_\alpha$$

and $q \Vdash s' \in \dot{H}$ for some $s' \geq \bar{s}$. Let $Bad(p, \dot{H}, D)$ be the set of bad δ .

Now for all such p, \dot{H}, D , $Bad(p, \dot{H}, D)$ is nonstationary. Otherwise, we could define a function \dot{f} (a promise), take p^* , adding the promise, and then find q .

($q \in D$, $q \subset p^* = p$ with \dot{f} as an extra promise) $\bar{s} \Vdash \forall \alpha \in \dot{Y}(\dot{f}, q, p^*, s) s \in S(\alpha_q) = \bar{s}|_{\alpha_q}$ then

$$[q, s] \setminus [p^*, s|_{\alpha_{p^*}}] = [q, s] \setminus [p, s|_{\alpha_p}] \subset \dot{A}_\alpha \setminus \dot{f}(\alpha) = \dot{A} \setminus F(\alpha),$$

a contradiction.

Theorem 45. *The forcing P is proper and preserves S*

Proof. Fix a countable $X \prec H((2^{|P|})^+)$, with $P, S \in X$. Let $p \in X \cap P$. Let $\delta = X \cap \omega_1$. Fix $\bar{s} \in S(\delta)$. We let I and II cooperate to pick a (P, X) -generic $q \in P$ such that \bar{s} is $X[q]$ -generic for S .

Let π be a winning strategy for Player I in $\Gamma(\delta, X \cap P, p)$. We need to show that if $D \in X$ is a dense open subset of P , e is a finite subset of δ , $p' \in P \cap X$ and \dot{H} is a P -name in X for a maximal antichain in S , then there II is a condition $p'' \leq p$ in X such that $p'' \in D$, $p'' \Vdash s' \in \dot{H}$ for some $s' \geq \bar{s}$, and for all $s \in S(\delta)$ with $s \restriction [\alpha_{p'}, \delta) = a_\delta \restriction [\alpha_{p'}, \delta)$,

$$s \Vdash [p'', s] \setminus [p', s \restriction \alpha_{p'}] \subset \dot{A}_\delta \setminus e.$$

picks conditions meeting dense sets and making \bar{s} meet new dense sets in S . For each (p_n, D_n, ξ_n) II sees and $\alpha < \delta$ such that can do her job missing an infinite set, so miss I 's play F_n . If f is a promise, α is banned by f if....

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