Dynamics of continued fractions and kneading sequences of unimodal maps

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Credits

Results contained in joint works with C. Carminati (Pisa), C. Bonanno (Pisa), S. Isola (Camerino)

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Thanks to C. McMullen

Motivation

Sullivan's dictionary

limit sets of subgroups of $PSL_2(\mathbb{C})$	$\Leftrightarrow$	Julia sets of rational maps

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limit sets of semigroups of $SL_2(\mathbb{Z})$	$\Leftrightarrow$	Julia sets of rational maps

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Question: Are there continuous families of objects on the left side?

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Question: How do their parameter spaces look like?

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The *bifurcation locus* is the set of locally unstable parameters ($\cong$ real boundary of the Mandelbrot set)

Numbers of bounded type

Consider the set

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- $m(\mathcal{B}_N) = 0$
- $\lim_{N \to \infty} \text{H.dim } \mathcal{B}_N = 1$ [Jarnik]

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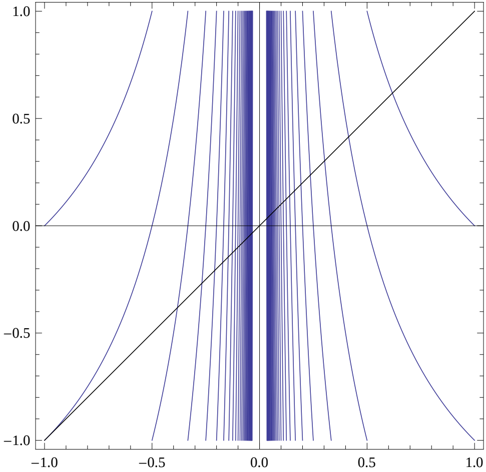
The map $t \mapsto \mathcal{B}(t)$ is locally constant near every $t \in \mathbb{Q}$.

α-continued fractions (Nakada '81)



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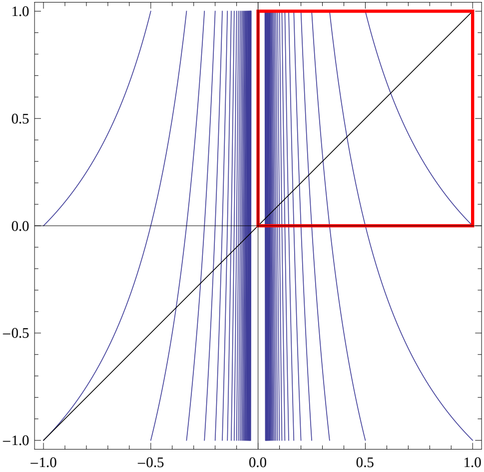

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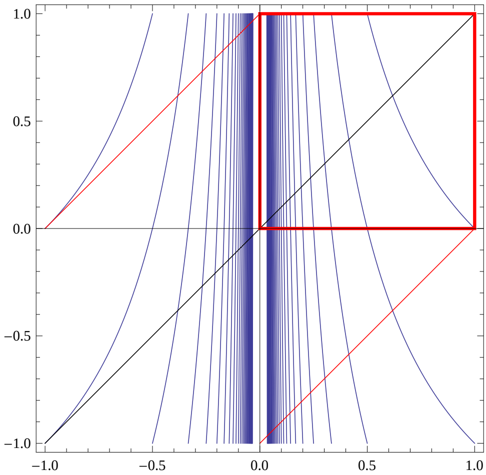

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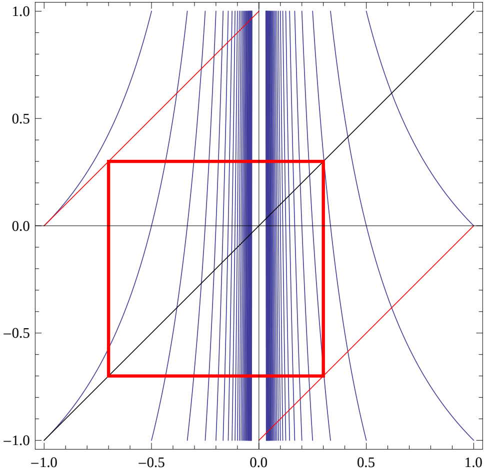

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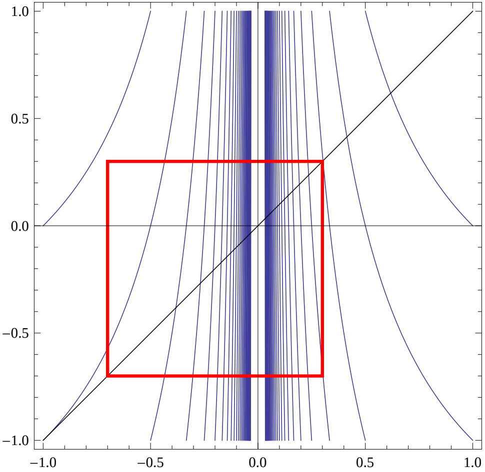

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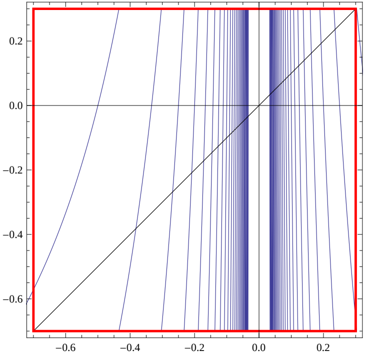
$$G_\alpha(x) := \frac{1}{|x|} - c_\alpha(x), \quad c_\alpha(x) := \lfloor \frac{1}{|x|} + 1 - \alpha \rfloor.$$



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and generate the α-continued fraction expansion:

$$x = \cfrac{\epsilon_{1,\alpha}}{c_{1,\alpha} + \cfrac{\epsilon_{2,\alpha}}{c_{2,\alpha} + \ddots}} \quad c_{n,\alpha} \in \mathbb{N}^+, \epsilon_{n,\alpha} \in \{\pm 1\}$$

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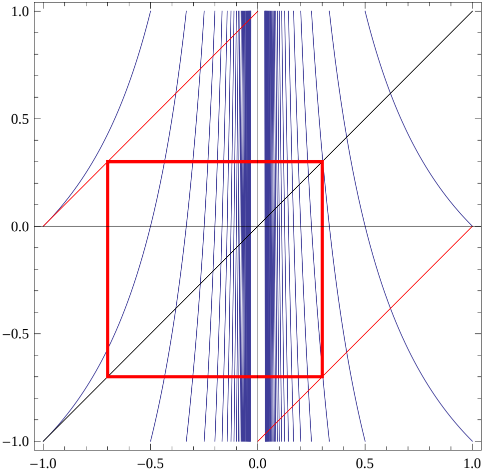

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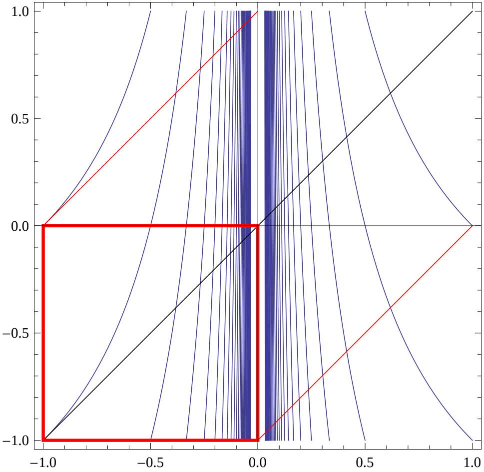

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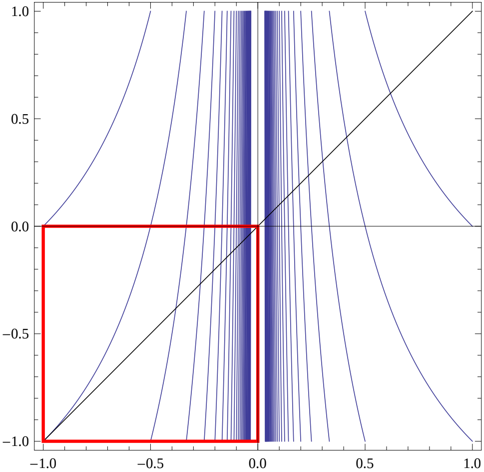

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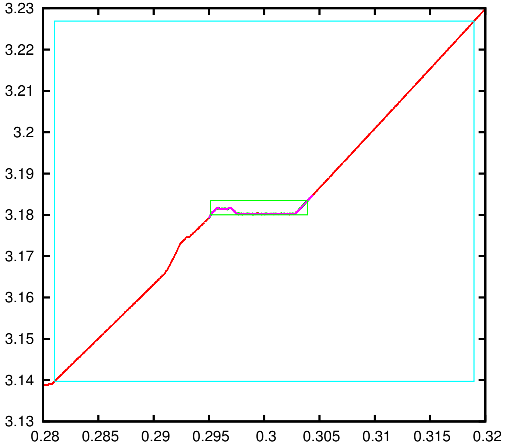
Conjecture(Nakada-Natsui): h is locally monotone almost everywhere.

The entropy is not monotone!



Luzzi-Marmi, ’08

The exceptional set $\mathcal{E}$

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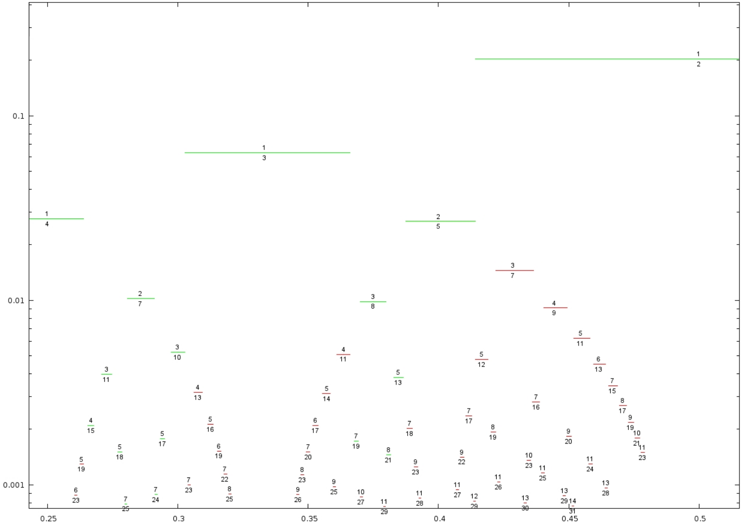
- $\mathcal{E}$ does not contain *any* rational number

Quadratic intervals



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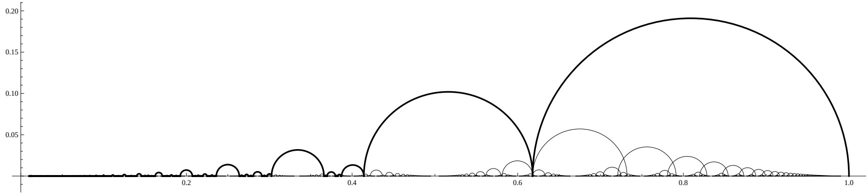
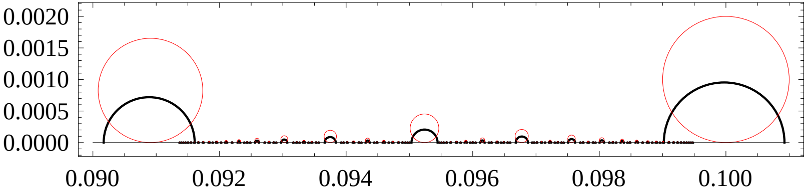


$\mathcal{E}$ vs horoballs



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Theorem (Carminati-T, '10)

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Corollary

Nakada-Natsui's conjecture holds.

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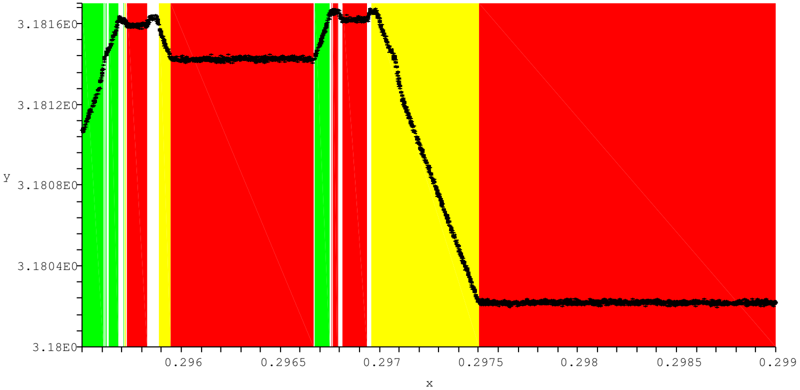

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Theorem (Carminati-T, 2011)

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- *The map* $t \mapsto \mathcal{B}(t)$ *is locally constant precisely on the complement of* $\mathcal{E}$
- *The dimension function* $t \mapsto \text{H.dim}\,\mathcal{B}(t)$ *is continuous decreasing.*

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$$a = [0; A^{\pm}] \mapsto I_a := (\alpha^-, \alpha^+), \quad \alpha^{\pm} := [0; \overline{A^{\pm}}].$$

The interval $I_a := (\alpha^-, \alpha^+)$ will be called the *quadratic interval* generated by $a \in \mathbb{Q} \cap (0,1)$.

Thickening $\mathbb{Q}$

$$\mathcal{M} = \bigcup_{a \in \mathbb{Q} \cap]0,1]} I_a.$$

- $\mathcal{M}$ is an open neighbourhood of $\mathbb{Q} \cap]0,1]$;
- the connected components of $\mathcal{M}$ are quadratic intervals;

The *exceptional set*

$$\mathcal{E} := [0,1] \setminus \mathcal{M} = [0,1] \setminus \bigcup_{a \in \mathbb{Q} \cap]0,1]} I_a$$

is such that

- $|\mathcal{E}| = 0$;
- $\dim_{\mathcal{H}}(\mathcal{E}) = 1$;

Symbolic dynamics of unimodal maps

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$$i(x) = s_1 s_2 \dots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{if } f^{i-1}(x) < c_0 \\ 1 & \text{if } f^{i-1}(x) \geq c_0 \end{cases}$$

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The complexity of the orbits of a unimodal map is encoded by its *kneading sequence*

$$K(f) = i(f(c_0)) \in \{0,1\}^{\mathbb{N}}$$

Symbolic dynamics of unimodal maps

Let $f : [0, 1] \to [0, 1]$ be a smooth map, F is called *unimodal* if it has exactly one critical point $0 < c_0 < 1$ and $f(0) = f(1) = 0$.

The *itinerary* of a point $x \in [0, 1]$ with f is the sequence

$$i(x) = s_1 s_2 \dots \quad \text{with} \quad s_i = \begin{cases} 0 & \text{if } f^{i-1}(x) < c_0 \\ 1 & \text{if } f^{i-1}(x) \geq c_0 \end{cases}$$

The complexity of the orbits of a unimodal map is encoded by its *kneading sequence*

$$K(f) = i(f(c_0)) \in \{0, 1\}^{\mathbb{N}}$$

Theorem (Milnor-Thurston '77)

The kneading sequence determines the topological entropy

The set of all kneading sequences Λ

Using the kneading sequence one can produce a *kneading invariant* τ_f

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Proposition

The set of all kneading invariants of all real quadratic polynomials is

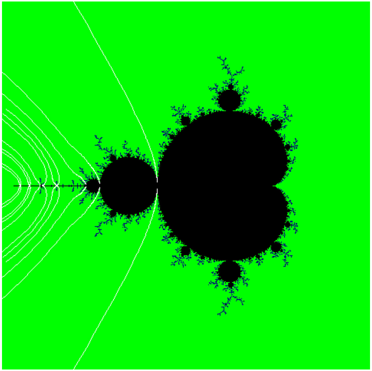
$$\Lambda := \{x \in [0,1] : T^k(x) \leq x \; \forall k \in \mathbb{N}\}$$

where T is the classical tent map.

Λ vs Mandelbrot

$$\Lambda := \{x \in [0,1] : T^k(x) \leq x \; \forall k \in \mathbb{N}\}$$

Λ corresponds to the set of external rays which 'land' on the bifurcation locus of the real quadratic family, i.e. the real slice of the Mandelbrot set.



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Theorem (Bonanno-Carminati-Isola-T, '10)

The sets $\Lambda \setminus \{0\}$ *and* $\mathcal{E}$ *are homeomorphic.*

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is an orientation-reversing homeomorphism which maps $\mathcal{E}$ *onto* $\Lambda \setminus \{0\}$.

Corollary

$$\text{H.dim}\ \mathcal{E} = 1 \Leftrightarrow \text{H.dim}\ (\partial M \cap \mathbb{R}) = 1$$

[Zakeri, Jakobson]

Minkowski's question mark function

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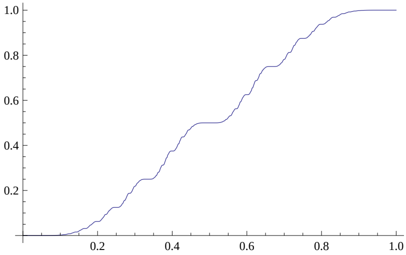


Further developments

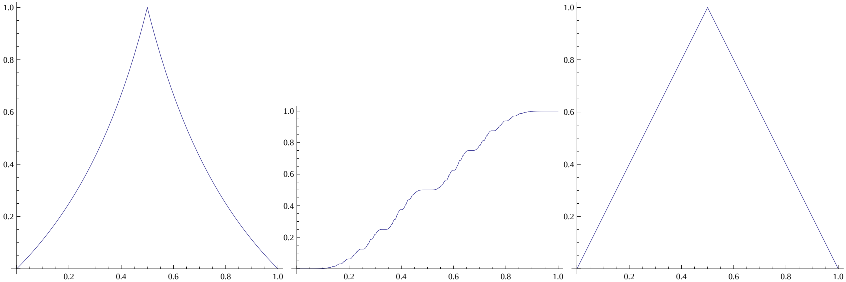
- Correspondence between $\mathcal{B}(t)$ and sets of external rays landing on real slice of Julia sets
- Local Hausdorff dimension of $\mathcal{E}$ vs dimension of individual $\mathcal{B}(t)$
- Renormalization for α-continued fractions

The end

Thank you!

From Farey to the tent map, via ?

Minkowski questionmark function conjugates the Farey map with the tent map



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