

**RANDOM WALKS ON WEAKLY HYPERBOLIC GROUPS**  
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1. INTRODUCTION TO RANDOM WALKS

1.1. **Basic examples.** Consider a drunkard who moves in a city by tossing coins to decide whether to go North, South, East or West: can he/she get back home?

It depends on the topography of the city.

**Example 1: Squareville**

In Squareville, blocks form a square grid. What is the probability of coming back to where you start? Let us first consider the easier case where your world is just a line, and you can only go in two directions: left or right.

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**Definition 1.1.** A random walk  $(w_n)$  on  $X$  is recurrent if for any  $x \in X$ , the probability that  $w_n = x$  infinitely often is 1:

$$\mathbb{P}(w_n = x \text{ i.o.}) = 1$$

Otherwise it is said to be transient.

Let us denote as  $p^n(x, y)$  the probability of being at  $y$  after  $n$  steps starting from  $x$ .

**Lemma 1.2.** Let  $m = \sum_{n \geq 1} p^n(0, 0)$  be the “average number of visits to 0”. Then the random walk is recurrent iff  $m = \infty$ .

**Exercise.** Prove the Lemma.

Now, what is the probability of going back to where you start after  $N$  steps? If  $N$  is odd, the probability is zero, but if  $N = 2n$  you get

$$p^{2n}(0, 0) = \frac{1}{2^{2n}} \binom{2n}{n} \quad (\text{choose } 2n \text{ ways to go right})$$

Is  $\sum_{n \geq 1} \frac{1}{2^{2n}} \binom{2n}{n}$  convergent?

Apply Stirling’s Formula:  $n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$

$$\frac{1}{2^{2n}} \binom{2n}{n} \sim \frac{1}{2^{2n}} \frac{\sqrt{n} \left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{n} \left(\frac{n}{e}\right)^n\right)^2} = \frac{1}{\sqrt{n}}$$

$\therefore$  our RW is recurrent.

Now, let us go to Squareville, i.e. the case of the 2-dimensional grid. Now we have 4 directions to choose from. One can check that

$$p^{2n}(0, 0) = \frac{1}{4^{2n}} \binom{2n}{n}^2 \sim \frac{1}{n} \quad (\text{match left \& right and match up and down})$$

hence the random walk is recurrent.

In general, one has the following.

**Theorem 1.3** (Polya). The simple random walk on  $\mathbb{Z}^d$  is recurrent iff  $d = 1, 2$ .

That is, “a drunk man will get back home, but a drunk bird will get lost”.

**Exercise.** Prove Polya’s theorem for  $d = 3$ . Moreover, for the simple random walk on  $\mathbb{Z}^d$  one can show that  $p^{2n}(0, 0) \approx n^{-\frac{d}{2}}$ .

**Example 2: Tree City**

In Tree City, the map has the shape of a 4-valent tree.

**Theorem 1.4.** *The simple random walk on a 4-valent tree is transient.*

We want to look at  $d_n =$  “distance of the  $n^{\text{th}}$  step of the RW from the origin”.

If you give the position of the  $n^{\text{th}}$  step, then finding  $d_{n+1}$  is as follows: if  $d_n > 0$  then

$$d_{n+1} = \begin{cases} d_n + 1 & \text{with } \mathbb{P} = \frac{3}{4} \\ d_n - 1 & \text{with } \mathbb{P} = \frac{1}{4} \end{cases}$$

and if  $d_n = 0$  then

$$d_{n+1} = d_n + 1$$

$$\therefore \mathbb{E}(d_{n+1} - d_n) \geq \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\therefore \mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2}$$

If we know that  $\lim_{n \rightarrow \infty} \frac{d_n}{n}$  exists almost surely and is constant, then  $\lim_{n \rightarrow \infty} \frac{d_n}{n} \geq \frac{1}{2}$  a.s. as  $\mathbb{E}\left(\frac{d_n}{n}\right) \geq \frac{1}{2} \Rightarrow$  RW is transient

**Exercise.** A *radially symmetric tree* of valence  $(a_1, a_2, \dots)$  is a tree where all vertices at distance  $n$  from the base point have exactly  $a_{n-1}$  children. Prove that the simple random walk on a radially symmetric tree  $(a_1, a_2, \dots)$  is transient iff

$$\sum_{n \geq 1} \frac{1}{a_1 \cdot a_2 \cdots a_n} < \infty$$

**1.2. General setup.** Let  $G$  be a group and  $(X, d)$  a metric space. The *isometry group of  $X$*  is the group of elements which preserve distance:

$$\text{Isom}(X) = \{f : X \rightarrow X : d(x, y) = d(f(x), f(y)) \text{ for all } x, y \in X\}$$

**Definition 1.5.** A group action of  $G$  on  $X$  is a homomorphism

$$\rho : G \rightarrow \text{Isom}(X).$$

The first example is the group of reals acting on themselves by translations: here  $X = \mathbb{R}$ ,  $G = \mathbb{R}$  and the action  $\rho : \mathbb{R} \rightarrow \text{Isom}(\mathbb{R})$  is given by  $\rho(t) : x \mapsto x + t$ .

Let us now consider a probability measure  $\mu$  on  $G$ . Then we can draw a sequence  $(g_n)$  of elements of  $G$ , independently and with distribution  $\mu$ . The sequence  $(g_n)$  is called the sequence of *increments*, and we are interested in the products

$$w_n := g_1 \cdots g_n$$

The sequence  $(w_n)$  is called a *sample path* for the random walk.

More formally, the *space of increments* is the infinite product space  $(G^{\mathbb{N}}, \mu^{\mathbb{N}})$ , and its elements are sequences  $(g_n)$  of elements of  $G$ . Then consider the map  $G^{\mathbb{N}} \rightarrow G^{\mathbb{N}}$

$$(g_n) \mapsto (w_n)$$

defined as where  $w_n = g_1 g_2 \dots g_n$  and define as the *sample space* as the space  $(\Omega, \mathbb{P})$  where  $\Omega = G^{\mathbb{N}}$  and  $\mathbb{P}$  is the pushforward of  $\mu^{\mathbb{N}}$  by the above map. Elements of  $\Omega$  are denoted as  $(w_n)$  and are called *sample paths*.

If you fix a basepoint  $x \in X$  (where  $X$  is the metric space) you can look at the sequence  $(w_n \cdot x) \subseteq X$ .

### Examples

- (1) The group  $G = \mathbb{Z}$  acts by translations on  $X = \mathbb{R}$ . Let us take  $\mu = \frac{\delta_{+1} + \delta_{-1}}{2}$ , i.e. one moves forward by 1 with probability  $\frac{1}{2}$  and moves backward by 1 with probability  $\frac{1}{2}$ . This is the simple random walk on  $\mathbb{Z}$  as previously discussed.
- (2) The same holds for  $G = \mathbb{R}^d$  or  $G = \mathbb{Z}^d$  acting by translations on  $X = \mathbb{R}^d$ . For  $d = 2$  and  $\mu = \frac{1}{4}(\delta_{(1,0)} + \delta_{(-1,0)} + \delta_{(0,1)} + \delta_{(0,-1)})$  you get the simple random walk on  $\mathbb{Z}^2$  (i.e. the random walk on Squareville).
- (3)  $X = 4$ -valent tree

$$G = \mathbb{F}_2 = \{ \text{reduced words in the alphabet } \{a, b, a^{-1}, b^{-1}\} \}$$

Here, *reduced* means that there are no redundant pairs, i.e. there is no  $a$  after  $a^{-1}$ , no  $a^{-1}$  after  $a$ , no  $b$  after  $b^{-1}$ , and no  $b^{-1}$  after  $b$ .

$$\mu = \frac{1}{4}(\delta_a + \delta_{a^{-1}} + \delta_b + \delta_{b^{-1}})$$

More generally, given a finitely generated group we can define its *Cayley graph*:

**Definition 1.6.** *Given a group  $G$  finitely generated by a set  $S$ , the Cayley graph  $\Gamma = \text{Cay}(G, S)$  is a graph whose vertices are the elements of  $G$  and there is an edge  $g \rightarrow h$  ( $g, h \in G$ ) if  $h = gs$  where  $s \in S$ .*

Often one takes  $S = S^{-1}$ , so that  $\text{Cay}(G, S)$  is an undirected graph.

**Definition 1.7.** *Given a finitely generated group  $G$  and a finite generating set  $S$ , we define the word length of  $g \in G$  as*

$$\|g\| := \min\{k : g = s_1 s_2 \dots s_k, s_i \in S \cup S^{-1}\}.$$

Moreover, we define the word metric or word distance between  $g, h \in G$  as

$$d(g, h) := \|g^{-1}h\|.$$

Note that with this definition left-multiplication by an element of  $G$  is an isometry for the word metric: for any  $h \in G$ ,  $d(gh_1, gh_2) = d(h_1, h_2)$ .

If we let  $G = \mathbb{F}_2$ ,  $S = \{a, b\}$ , then  $\text{Cay}(\mathbb{F}_2, S)$  is the 4-valent tree. On the other hand, if  $G = \mathbb{Z}^2$ ,  $S = \{(1, 0), (0, 1)\}$  then  $\text{Cay}(\mathbb{Z}^2, S)$  is the square grid.

Note that  $\text{Cay}(\mathbb{Z}^2, S)$  has loops, since e.g.  $(0, 1) + (-1, 0) + (0, -1) + (1, 0) = (0, 0)$ , while we do not have a corresponding loop in  $\mathbb{F}_2$  as the element in  $\mathbb{F}_2$  that would correspond to the loop is  $ab^{-1}a^{-1}b$  and is not trivial.

- (4) Consider  $G = SL_2(\mathbb{R}) = \{A \in M_2 : \det A = 1\}$  which acts on the hyperbolic plane  $X = \mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$  by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = \frac{az + b}{cz + d}.$$

The group  $G$  acts by isometries for the *hyperbolic metric*  $ds = \frac{dx}{y}$ . Let  $A, B \in SL_2(\mathbb{R})$ ,  $\mu = \frac{1}{4}(\delta_A + \delta_{A^{-1}} + \delta_B + \delta_{B^{-1}})$ . The boundary of  $\mathbb{H}$  is  $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ . As we will see later, random walks of this type converge almost surely to the boundary. Equivalently, one can also use the Poincaré disc model. The disc has a natural topological boundary, i.e. the circle.

### Questions

- (1) Does a typical sample path escape to  $\infty$  or it comes back to the origin infinitely often?
- (2) If it escapes, does it escape with “positive speed”?

**Definition 1.8.** We define the drift or speed or rate of escape of the random walk to be the limit

$$L := \lim_{n \rightarrow \infty} \frac{d(w_n x, x)}{n} \quad (\text{if it exists})$$

A measure  $\mu$  on  $G$  has *finite first moment* on  $X$  if for some (equivalently, any)  $x \in X$

$$\int_G d(x, gx) d\mu(g) < +\infty.$$

**Lemma 1.9.** If  $\mu$  has finite first moment, then there exists a constant  $L \in \mathbb{R}$  such that for a.e. sample path

$$\lim_{n \rightarrow \infty} \frac{d(w_n x, x)}{n} = L.$$

*Proof.* For any  $x \in X$ , the function  $a(n, \omega) := d(x, w_n(\omega)x)$  is a subadditive cocycle, because

$$d(x, w_{n+m}(\omega)x) \leq d(x, w_n(\omega)x) + d(w_n(\omega)x, w_{n+m}(\omega)x) =$$

and since  $w_n$  is an isometry

$$= d(x, w_n(\omega)x) + d(x, g_{n+1} \dots g_{n+m}x) = d(x, w_n(\omega)x) + d(x, w_m(T^n \omega)x)$$

where  $T$  is the shift on the space of increments, hence the claim follows by Kingman's subadditive ergodic theorem (Theorem 5.3).  $\square$

- (3) Does a sample path track geodesics in  $X$ ? How closely?
- (4) If  $X$  has a topological boundary  $\partial X$ , does a typical sample path converge to  $\partial X$ ?

**Definition 1.10.** *If so, define the hitting measure  $\nu$  on  $\partial X$  as*

$$\nu(A) = \mathbb{P}(\lim_{n \rightarrow \infty} g_n x \in A)$$

for any  $A \subset \partial X$ .

- (5) What are the properties of hitting measure? Is it the same as the geometric measure? For example, is it the same as the Lebesgue measure?
- (6) What is the boundary of a group? This leads to the notion of *Poisson boundary*.

## 2. GROMOV HYPERBOLIC SPACES

### Hyperbolic spaces.

Let  $(X, d)$  be a geodesic, metric space, and let  $x_0 \in X$  be a basepoint.

Define the *Gromov product* of  $x, y$  as :

$$(x, y)_{x_0} := \frac{1}{2} (d(x_0, x) + d(x_0, y) - d(x, y))$$

**Definition 2.1.** *The geodesic metric space  $X$  is  $\delta$ -hyperbolic if geodesic triangles are  $\delta$ -thin.*

We will use the notation

$$A = B + O(\delta)$$

to mean that there exists  $C$ , which depends only on  $\delta$ , for which  $|A - B| \leq C$ .

If  $X$  is  $\delta$ -hyperbolic  $\implies (x, y)_{x_0} = d(x_0, [x, y]) + O(\delta)$

**Example 2.2.** *The following are  $\delta$ -hyperbolic spaces:*

$$X = \mathbb{R} \checkmark$$

$$X = \text{tree} \checkmark$$

$$G = \mathbb{F}_2, X = \text{Cay}(\mathbb{F}_2, S) \checkmark$$

**Definition 2.3.** *A group  $G$  is word hyperbolic if there is a finite set  $S$  of generators such that  $\text{Cay}(G, S)$  is  $\delta$ -hyperbolic.*

Note: The fact that  $G$  is word hyperbolic does not depend on the choice of  $S$  (**Exercise:** why?).

**Definition 2.4.** *The action of  $G$  on  $X$  is properly discontinuous if for any  $x \in X$ ,  $\exists U \ni x$  such that  $\#\{g \in G : gU \cap U \neq \emptyset\}$  is finite.*

Let us now consider

$G$  countable  $< \text{Isom}(\mathbb{H}^n)$  (where  $\mathbb{H}^n$  is an  $n$ -dimensional hyperbolic space)

If the action of  $G$  on  $\mathbb{H}^n$  is properly discontinuous and cocompact, then  $G$  is word hyperbolic. This is a special case of the following:

**Lemma 2.5** (Švarc-Milnor). *If  $G$  acts properly discontinuously and cocompactly on a  $\delta$ -hyperbolic space, then  $G$  is word hyperbolic.*

**Example 2.6.** *Let  $S =$  surface of genus  $g \geq 2$ . Then  $\pi_1(S)$  is word hyperbolic. In fact,  $\tilde{S} \simeq \mathbb{D} \simeq \mathbb{H}^2$ , and there is a regular  $4g$ -gon in  $\mathbb{H}^2$  with angles  $\frac{2\pi}{4g}$ . Then  $\mathbb{H}^2 / G = S$  where  $G = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle = \pi_1(S)$ , and the action of  $G$  on  $\mathbb{H}^2$  is properly discontinuous and cocompact.*

Note: if  $G < \text{Isom}(\mathbb{H}^3)$  which acts properly discontinuously but not cocompactly, then  $G$  need not be word hyperbolic (it may contain  $\mathbb{Z}^2$ ). The same is true for  $n \geq 3$  (how about  $n = 2$ ?)

### The mapping class group.

Let  $S$  be a closed, orientable, surface of genus  $g \geq 2$ . The *mapping class group* of  $S$  is

$$\text{Mod}(S) := \text{Homeo}^+(S) / \text{isotopy}$$

and is a countable, finitely generated group.

Note that  $\text{Mod}(S)$  is *not* word hyperbolic. In fact, if you fix a curve  $\alpha$  on  $S$ , you can define a *Dehn twist*  $D_\alpha$  around this curve. Then, if  $\alpha, \beta$  are disjoint then  $\langle D_\alpha, D_\beta \rangle = \mathbb{Z}^2$ .

However, the mapping class group *does* act on a  $\delta$ -hyperbolic space (but this space is *not* proper!).

If  $S$  is a topological surface of finite genus  $g$ , possibly with finitely many boundary components, then the *curve graph*  $\mathcal{C}(S)$  is a graph whose vertices are isotopy classes of essential<sup>1</sup>, simple closed curves on  $S$ , and there is an edge  $\alpha \rightarrow \beta$  if  $\alpha$  and  $\beta$  have disjoint representatives.

**Theorem 2.7** (Masur-Minsky [MM99]). *The curve graph is  $\delta$ -hyperbolic.*

In fact, one can also define the *curve complex* by considering the simplicial complex where every  $k$ -simplex represents a set of  $k$  disjoint curves on the surface. The curve graph is the 1-skeleton of the curve complex, and it is quasi-isometric to it. Thus, for most purposes, it is enough to work with the curve graph.

**Exercise.** Consider a closed surface of genus  $g \geq 2$ . Prove that the curve graph has diameter  $\geq 2$ . In fact, prove that it has infinite diameter.

**Exercise.** Consider a surface of genus  $g$  with  $n$  punctures. Figure out for what values of  $g, n$  the curve graph is empty, and for what values of  $n$  it is disconnected. In the latter case, think of how to modify the definition in order to obtain a connected space.

### Outer automorphisms of the free group.

Let  $F_n$  be a free group of rank  $n$ , and let  $G = \text{Out}(F_n) = \text{Aut}(F_n)/\text{Inn}(F_n)$  the group of outer automorphisms of  $F_n$ . Then for  $n \geq 2$ ,  $G$  is not a word hyperbolic group but it acts on several non-proper hyperbolic spaces.

In particular, the *free factor complex*  $\mathcal{FF}(F_n)$  is a countable graph whose vertices are conjugacy classes of proper free factors of  $F_n$ , and simplices are determined by chains of nested free factors. (A *free factor* is a subgroup  $H$  of  $F_n$  such that there exists another subgroup  $K$  so that  $F_n = H \star K$ ). The graph  $\mathcal{F}$  is hyperbolic by Bestvina-Feighn [BF14]. Another hyperbolic space on which  $\text{Out}(F_n)$  acts is the *free splitting complex*  $\mathcal{FS}(F_n)$ .

An element of  $\text{Out}(F_n)$  is *fully irreducible* if for all proper free factors  $F$  of  $F_n$  and all  $k > 0$ ,  $f^k(F)$  is not conjugate to  $F$ . An element is loxodromic on  $\mathcal{FF}(F_n)$  if and only if it is fully irreducible, and all fully irreducible elements satisfy the WPD property.

**Exercise.** What is  $\text{Out}(F_2)$ ? How about its corresponding free factor complex?

### Right-angled Artin groups.

Let  $\Gamma$  be a finite graph. Define the right-angled Artin group  $A(\Gamma)$  as

$$A(\Gamma) = \langle v \in V(\Gamma) : vw = wv \text{ if } (v, w) \in E(\Gamma) \rangle$$

---

<sup>1</sup>Recall that a curve on a surface is *essential* if it is not homotopic to either a point or a boundary component.



Right-angled Artin groups act on  $X =$  extension graph where vertices are conjugacy classes of elements of  $V(\Gamma)$ , and there is an edge between  $v^g$  and  $u^h$  iff they commute. Acylindricity of the action is due to Kim-Koberda.

**Relatively hyperbolic groups.**

Let  $H$  be a finitely generated subgroup of a finitely generated group  $G$ , and fix a finite generated set  $S$  for  $G$ . Then consider the Cayley graph  $X = \text{Cay}(G, S)$ , and construct a new graph  $\widehat{X}$  as follows. For each left coset  $gH$  of  $H$  in  $G$ , add a vertex  $v(gH)$  to  $\widehat{X}$ , and add an edge from  $v(gH)$  to each vertex representing an element of  $gH$ .

The group  $G$  is *hyperbolic relative to  $H$*  if the coned-off space  $\widehat{X}$  is a  $\delta$ -hyperbolic space;  $\widehat{X}$  is not proper as long as  $H$  is infinite.

**The Cremona group.**

The *Cremona group* is the group of birational transformations of the complex projective plane  $\mathbb{C}\mathbb{P}^2$ . That is, every element is given by

$$f([x : y : z]) := [P(x, y, z) : Q(x, y, z) : R(x, y, z)]$$

where  $P, Q, R$  are three homogenous polynomials of the same degree, without common factors. The common degree of  $P, Q, R$  is called the *degree* of  $f$ .

The Cremona group acts on the *Picard-Manin space*, which is given by taking the cohomology of all possible blowups of  $\mathbb{P}^2$ , and preserves a quadratic form of signature  $(1, \infty)$ . Hence, the Cremona group acts by isometries on a hyperboloid  $\mathbb{H}_{\mathbb{P}^2}$  in the Picard-Manin space, which is indeed a non-proper  $\delta$ -hyperbolic space. For details, see [CL13], [MT2].

Moreover, an element is WPD if it is not conjugate to a monomial map, i.e. a map which is in affine coordinates of the form  $f(x, y) := (x^a y^b, x^c y^d)$  where  $ad - bc \neq 0$ .

**Exercise.** Read the definition of the Picard-Manin space in the appendix.

**Exercise.** Find an example of two Cremona transformations  $f, g$  such that  $\deg f \circ g \neq \deg f \cdot \deg g$ .

**2.1. The Gromov boundary.**

**Definition 2.8.** A metric space is *proper* if closed balls (i.e. sets of the form  $\{y \in X : d(x, y) \leq r\}$ ) are compact.

Let  $X$  be a  $\delta$ -hyperbolic metric space. If  $X$  is proper then we can give the following definition of the boundary of  $X$ . Fix a base point  $x_0 \in X$ . We declare two geodesic rays  $\gamma_1, \gamma_2$  based at  $x_0$  to be *equivalent* if  $\sup_{t \geq 0} d(\gamma_1(t), \gamma_2(t)) < \infty$  and we denote this as  $\gamma_1 \sim \gamma_2$ .

We define the *Gromov boundary* of  $X$  as

$$\partial X := \{\gamma \text{ geodesic rays based at } x_0\} / \sim$$

**Example 2.9.** *Examples of Gromov boundaries.*

- $X = \mathbb{R}$  and  $\partial X = \{-\infty, +\infty\}$ .
- $X = \text{ladder}$  and  $\partial X = \{-\infty, +\infty\}$ .

In general (if  $X$  is not necessarily proper) we define the boundary using sequences.

A sequence  $(x_n) \subset X$  is a *Gromov sequence* if  $\liminf_{m,n \rightarrow \infty} (x_n \cdot x_m)_{x_0} = \infty$ . Two Gromov sequences  $(x_n), (y_n)$  are *equivalent* if  $\liminf_{n \rightarrow \infty} (x_n, y_n)_{x_0} = \infty$ . In general we define the boundary of  $X$  as

$$\partial X := \{(x_n) \text{ Gromov sequence}\} / \sim$$

where  $\sim$  denotes equivalence of Gromov sequences.

**Theorem 2.10.**  *$\partial X$  is a metric space.*

In order to define the metric, let  $\eta, \xi \in \partial X$ . Then  $\eta = [x_n], \xi = [y_n]$  for two Gromov sequences  $(x_n), (y_n)$ . Then one defines

$$(\eta \cdot \xi)_{x_0} := \sup_{x_n \rightarrow \eta, y_n \rightarrow \xi} \liminf_{m,n} (x_m \cdot y_n)_{x_0}$$

Pick  $\epsilon > 0$ , and set  $\rho(\xi, \eta) := e^{-\epsilon(\eta \cdot \xi)_{x_0}}$ . This is not yet a metric (no triangle inequality). To get an actual metric, you need to take

$$d(\xi, \eta) := \inf \sum_{i=1}^{n-1} \rho(\xi_i, \xi_{i+1})$$

where the inf is taken along all finite chains  $\xi = \xi_0, \xi_1, \dots, \xi_{n-1}, \eta = \xi_n$ .

**Lemma 2.11.**  *$\exists C = C(\epsilon)$  such that*

$$C\rho(\xi, \eta) \leq d(\xi, \eta) \leq \rho(\xi, \eta) \quad \forall \xi, \eta \in \partial X$$

If  $X$  is proper, then  $\partial X$  is a compact metric space, but if  $X$  is not proper, then  $\partial X$  need not be compact.

**Example 2.12.**  $X = \mathbb{N} \times \mathbb{R}^{\geq 0} / (n, 0) \sim (m, 0)$ . Then  $\partial X \simeq \mathbb{N}$  is not compact.

**2.2. Random walks on weakly hyperbolic groups.** Let  $(X, d)$  be a geodesic,  $\delta$ -hyperbolic, separable metric space, and let  $G$  be a countable group of isometries of  $X$ .

**Definition 2.13.** Given an isometry  $g$  of  $X$  and  $x \in X$ , we define its translation length of  $g$  as

$$\tau(g) := \lim_{n \rightarrow \infty} \frac{d(g^n x, x)}{n}$$

where the limit is independent of the choice of  $x$  (why?).

**Lemma 2.14** (Classification of isometries of hyperbolic spaces). *Let  $g$  be an isometry of a  $\delta$ -hyperbolic metric space  $X$  (not necessarily proper). Then either:*

- (1)  $g$  has bounded orbits. Then  $g$  is called elliptic.
- (2)  $g$  has unbounded orbits and  $\tau(g) = 0$ . Then  $g$  is called parabolic.
- (3)  $\tau(g) > 0$ . Then  $g$  is called hyperbolic or loxodromic, and has precisely two fixed points on  $\partial X$ , one attracting and one repelling.

Given a measure  $\mu$  on a countable group, its *support* is the set of elements  $g \in G$  with  $\mu(g) > 0$ . We will denote as  $\Gamma_\mu$  or  $\text{sgr}(\mu)$  the semigroup generated by the support of  $\mu$ .

**Definition 2.15.** *Two loxodromic elements are independent if their fixed point sets are disjoint. A probability measure  $\mu \in P(G)$  is non-elementary if  $\text{sgr}(\mu)$  contains 2 independent hyperbolic elements.*

The main results we are going to discuss in these lectures are the following.

**Theorem 2.16** (Maher-Tiozzo [MT]). *Let  $G$  be a countable group of isometries of a (separable)  $\delta$ -hyperbolic metric space  $X$ , such that the semigroup generated by the support of  $\mu$  is non-elementary. Then:*

- (1) For a.e.  $(w_n)$  and every  $x \in X$

$$\lim_{n \rightarrow \infty} w_n x = \xi \in \partial X \text{ exists}$$

- (2)  $\exists L > 0$  s.t.

$$\liminf_{n \rightarrow \infty} \frac{d(w_n x, x)}{n} = L > 0$$

If  $\mu$  has finite 1<sup>st</sup> moment then

$$\lim_{n \rightarrow \infty} \frac{d(w_n x, x)}{n} = L > 0 \text{ exists a.s.}$$

- (3) For any  $\epsilon > 0$  we have

$$\mathbb{P}(\tau(w_n) \geq n(L - \epsilon)) \rightarrow 1$$

as  $n \rightarrow \infty$ .

**2.3. The Poisson boundary.** The well-known *Poisson representation formula* expresses a duality between bounded harmonic functions on the unit disk and bounded functions on its boundary circle. Indeed, bounded harmonic functions admit radial limit values almost surely, while integrating a boundary function against the Poisson kernel gives a harmonic function on the interior of the disk. This picture is deeply connected with the geometry of  $SL_2(\mathbb{R})$ ; then in the 1960's Furstenberg and others extended this duality to more general groups.

**Definition 2.17.** *A function  $f : G \rightarrow \mathbb{R}$  is  $\mu$ -harmonic if it satisfies the mean value property with respect to averaging using  $\mu$ ; that is, it*

$$f(g) = \sum_{h \in G} f(gh) \mu(h) \quad \forall g \in G.$$

*We denote the space of bounded,  $\mu$ -harmonic functions as  $H^\infty(G, \mu)$ .*

Following Furstenberg [Fu1, Fu2], a measure space  $(M, \nu)$  on which  $G$  acts is then a boundary if there is a duality between bounded,  $\mu$ -harmonic functions on  $G$  and  $L^\infty$  functions on  $M$ .

**Definition 2.18.** *Given a  $(G, \mu)$ -space  $(B, \nu)$  one defines the Poisson transform as*

$$\Phi(f)(g) := \int_B f dg\nu$$

*and the space  $(B, \nu)$  is called the Poisson boundary if the Poisson transform is an isomorphism between  $H^\infty(G, \mu)$  and  $L^\infty(B, \nu)$ .*

If  $\nu$  is the hitting measure for a random walk, then a fundamental question in the field is then whether the pair  $(\partial X, \nu)$  equals indeed the Poisson boundary of the random walk  $(G, \mu)$ , i.e. if all harmonic functions on  $G$  can be obtained by integrating a bounded, measurable function on  $\partial X$ .

**The WPD condition.** In order to obtain the Poisson boundary in the non-proper case, we need a weak notion of properness introduced by Bestvina and Fujiwara *weak proper discontinuity (WPD)*. Intuitively, an element is WPD if it acts properly on its axis. In formulas, an element  $g \in G$  is *WPD* if for any  $x \in X$  and any  $K \geq 0$  there exists  $N > 0$  such that

$$(1) \quad \#\{h \in G : d(x, hx) \leq K \text{ and } d(g^N x, hg^N x) \leq K\} < +\infty.$$

In other words, the finiteness condition is not required of all pairs of points in the space, but only of points along the axis of a given loxodromic element.

**Theorem 2.19** (Poisson boundary for WPD actions, [MT2]). *Let  $G$  be a countable group which acts by isometries on a  $\delta$ -hyperbolic metric space  $(X, d)$ , and let  $\mu$  be a non-elementary probability measure on  $G$  with finite logarithmic moment and finite entropy. Suppose that there exists at least one WPD element  $h$  in the semigroup generated by the support of  $\mu$ . Then the Gromov boundary of  $X$  with the hitting measure is a model for the Poisson boundary of the random walk  $(G, \mu)$ .*

The WPD condition also has a consequence for the normal closure.

**Theorem 2.20** (Normal closure, [MT2]). *Let  $G$  be a group of isometries of  $\delta$ -hyperbolic metric space  $(X, d)$ , and let  $\mu$  be a non-elementary, bounded, reversible probability measure on  $G$  such that  $\Gamma_\mu$  contains at least one WPD element. Then there exists  $k \geq 1$  such that:*

(1)

$$\mathbb{P}(\langle\langle w_n^k \rangle\rangle \text{ is free}) \rightarrow 1$$

(2) *For almost every sample path, the sequence*

$$(\langle\langle w_1^k \rangle\rangle, \langle\langle w_2^k \rangle\rangle, \dots, \langle\langle w_n^k \rangle\rangle, \dots)$$

*contains infinitely many different normal subgroups of  $G$ .*

### Applications

- (1) word hyperbolic groups
- (2) relatively hyperbolic groups
- (3) right-angled Artin groups
- (4) mapping class groups
- (5)  $\text{Out}(F_n)$
- (6) Cremona group

## 3. THE HOROFUNCTION BOUNDARY

**3.1. The horofunction boundary.** Pick a base point  $x_0 \in X$ . For any  $z \in X$  we define the function  $\rho_z : X \rightarrow \mathbb{R}$ :

$$\rho_z(x) := d(x, z) - d(x_0, z).$$

Then  $\rho_z(x)$  is 1-Lipschitz and  $\rho_z(x_0) = 0$ .

Consider space  $\text{Lip}_{x_0}^1(X) = \{f : X \rightarrow \mathbb{R} \text{ s.t. } |f(x) - f(y)| \leq d(x, y), f(x_0) = 0\}$

with the topology of pointwise convergence. Let us consider the map  $\rho : X \rightarrow \text{Lip}_{x_0}^1(X)$  given by

$$z \mapsto \rho_z.$$

**Definition 3.1.** *The horofunction compactification of  $(X, d)$  is the closure  $\overline{X}^h := \overline{\rho(X)}$  in  $\text{Lip}_{x_0}^1(X)$ .*

**Proposition 3.2.** *If  $X$  is separable, then the horofunction compactification  $\overline{X}^h$  is a compact metrizable space.*

*Proof.* If one picks  $h \in \text{Lip}_{x_0}^1(X)$ , then

$$|h(x)| \leq |h(x) - h(x_0)| \leq d(x, x_0)$$

hence  $\text{Lip}_{x_0}^1(X) \subset \otimes_{x \in X} [-d(x, x_0), d(x, x_0)]$

which is compact by Tychonoff's theorem. Since  $X$  is separable, then  $C(X)$  is second countable, hence  $\overline{X}^h$  is second countable. Thus  $\overline{X}^h$  is compact, Hausdorff, and second countable, hence metrizable. □

**Exercise.** Prove that  $C(X)$  is second countable and Hausdorff if  $X$  is separable. Prove that a Hausdorff, second countable, compact topological space is metrizable.

Define the action of  $G$  on  $\overline{X}^h$  as

$$g.h(z) := h(g^{-1}z) - h(g^{-1}z_0)$$

for all  $g \in G$  and  $h \in \overline{X}^h$ .

The action of  $G$  on  $X$  extends to an action by homeomorphisms on  $\overline{X}^h$ .

**Example 3.3.**  $X = \mathbb{R}$  with the euclidean metric, and  $x_0 = 0$ . Then all horofunctions for  $X$  are either:

- $\rho(x) = |x - p| - |p|$  for some  $p \in \mathbb{R}$ ;

or

- $\rho(x) = \pm x$ .

hence  $\partial^h X = \overline{X}^h \setminus X = \{-\infty, +\infty\}$ .

**Example 3.4.** In the hyperbolic plane  $X = \mathbb{H}^2$ , pick  $\xi \in \partial\mathbb{H}^2$  and consider a geodesic ray  $\gamma : [0, \infty) \rightarrow \mathbb{H}^2$  with  $\gamma(0) = x_0$  and  $\lim_{t \rightarrow +\infty} \gamma(t) = \xi$ . Then if  $z_n := \gamma(n)$  we get for any  $x \in \mathbb{H}^2$

$$h_\xi(x) = \lim_{z_n \rightarrow \xi} \rho_{z_n}(x) = \lim_{t \rightarrow \infty} (d(\gamma(t), x) - t)$$

is the usual definition of horofunction, and level sets are horoballs.

**Example 3.5.** Let  $X =$  "infinite tree" defined as  $X = \mathbb{Z} \times \mathbb{R}^{\geq 0} / (n, 0) \sim (m, 0)$ .

Then the Gromov boundary is  $\partial X = \mathbb{Z}$ .

On the other hand, if  $z_n = [(n, n)]$  then in the horofunction compactification one has  $\lim_n \rho_{z_n} = \rho_{z_0}$ . If you think about it, this is related to the fact that the set of infinite horofunctions is not closed.

**Proposition 3.6** (Classification of horofunctions). *Let  $h$  be a horofunction in  $\overline{X}^h$ , and let  $\gamma$  be a geodesic in  $X$ . Then there is a point  $p$  on  $\gamma$  such that the restriction of  $h$  to  $\gamma$  is equal to exactly one of the following two functions, up to bounded additive error:*

- either

$$h(x) = h(p) + d(p, x) + O(\delta)$$

- or

$$h(x) = h(p) + d_\gamma^+(p, x) + O(\delta)$$

where  $d^+$  is the oriented distance along the geodesic, for some choice of orientation of  $\gamma$ .

For any horofunction  $h \in \overline{X}^h$ , let us consider

$$\inf(h) := \inf_{y \in X} h(y).$$

**Definition 3.7.** *The set of finite horofunctions is the set*

$$\overline{X}_F^h := \{h \in \overline{X}^h : \inf h > -\infty\}$$

and the set of infinite horofunctions is the set

$$\overline{X}_\infty^h := \{h \in \overline{X}^h : \inf h = -\infty\}$$

The key geometric lemma relating the geometry of the horofunction boundary and the Gromov boundary is the following.

**Lemma 3.8.** *For each base point  $x_0 \in X$ , each horofunction  $h \in \overline{X}^h$  and each pair of points  $x, y \in X$  the following inequality holds:*

$$\min\{-h(x), -h(y)\} \leq (x, y)_{x_0} + O(\delta)$$

*Proof.* Let  $z \in X$ . Then one has, by the triangle inequality

$$(x \cdot z)_{x_0} = \frac{d_X(x_0, x) + d_X(x_0, z) - d_X(x, z)}{2},$$

which implies

$$(x \cdot z)_{x_0} \geq d_X(x_0, z) - d_X(x, z),$$

and by definition, the right hand side is equal to  $-\rho_z(x)$ , which gives

$$(x \cdot z)_{x_0} \geq -\rho_z(x).$$

Now, by  $\delta$ -hyperbolicity one has

$$(x \cdot y)_{x_0} \geq \min\{(x \cdot z)_{x_0}, (y \cdot z)_{x_0}\} - \delta,$$

hence, by combining it with the previous estimate,

$$(x \cdot y)_{x_0} \geq \min\{-\rho_z(x), -\rho_z(y)\} - \delta.$$

Since every horofunction is the pointwise limit of functions of type  $\rho_z$ , the claim follows.  $\square$

This has the following consequence.

**Lemma 3.9.** *Let  $(x_n) \subseteq X$  be a sequence of points, and  $h \in \overline{X}^h$  a horofunction. If  $h(x_n) \rightarrow -\infty$ , then  $(x_n)$  converges in the Gromov boundary, and*

$$\lim_{n \rightarrow \infty} x_n \in \partial X$$

*does not depend on choice of  $(x_n)$ .*

**Definition 3.10.** *The local minimum map  $\varphi : \overline{X}^h \rightarrow X \cup \partial X$  is defined as follows.*

- *If  $h \in \overline{X}_F^h$ , then define*

$$\varphi(h) := \{x \in X : h(x) \leq \inf h + 1\}$$

- *If  $h \in \overline{X}_\infty^h$ , then choose a sequence  $(y_n)$  with  $h(y_n) \rightarrow -\infty$  and set*

$$\varphi(h) := \lim_{n \rightarrow \infty} y_n$$

*be the limit point in the Gromov boundary.*

**Lemma 3.11.** *There exists  $K$ , which depends only on  $\delta$ , such that for each finite horofunction  $h$  we have*

$$\text{diam } \varphi(h) \leq K.$$

*Proof.* Let  $x, y \in \phi(h)$ , for some  $h \in \overline{X}^h$ , and consider the restriction of  $h$  along a geodesic segment from  $x$  to  $y$ . By Proposition 3.6, the restriction has at most one coarse local minimum: hence, since  $x$  and  $y$  are coarse local minima of  $h$ , the distance between  $x$  and  $y$  is universally bounded in terms of  $\delta$ .  $\square$

**Corollary 3.12.** *The local minimum map  $\varphi : \overline{X}^h \rightarrow X \cup \partial X$  is well-defined and  $G$ -equivariant.*

Note:  $\varphi$  is not continuous but  $\varphi|_{\overline{X}_\infty^h}$  is continuous. For instance, in the “infinite tree” case of Example 3.5, if  $z_n := (n, n)$  then  $\rho_{z_n} \rightarrow \rho_{x_0}$  but  $\phi(\rho_{z_n}) = z_n \not\rightarrow x_0$ .

#### 4. BOUNDARY MEASURES AND CONVERGENCE

The goal of this lecture is to prove the following convergence result.



**Theorem 4.1** ([MT], Theorem 4.1). *Let  $G$  be a countable group of isometries of a geodesic, separable,  $\delta$ -hyperbolic space  $X$  (not necessarily proper), and let  $\mu$  be a non-elementary probability measure on  $G$ . Then for each  $x_0 \in X$ , almost every sample path  $(w_n x_0)$  converges in  $X \cup \partial X$  to a point of the Gromov boundary  $\partial X$ .*

#### 4.1. Stationary measures.

**Definition 4.2.** *Let  $\mu$  be a probability measure on a group  $G$ , and let  $M$  be a metric space on which  $G$  acts by homeomorphisms. A probability measure  $\nu$  on  $M$  is  $\mu$ -stationary (or just stationary) if*

$$\int_G g\nu \, d\mu(g) = \nu$$

*The pair  $(M, \nu)$  is then called a  $(G, \mu)$ -space.*

Problem: Since  $\partial X$  need not be compact, you may not be able to find a stationary measure in  $P(\partial X)$ . Trick: Consider the horofunction compactification (which is always compact and metrizable).

**Lemma 4.3.**  *$P(\overline{X}^h)$  is compact, so it contains a  $\mu$ -stationary measure.*

**Proposition 4.4.** *Let  $M$  be a compact metric space on which the countable group  $G$  acts continuously, and  $\nu$  a  $\mu$ -stationary Borel probability measure on  $M$ . Then for  $\mathbb{P}$ -a.e. sequence  $(w_n)$  the limit*

$$\nu_\omega := \lim_{n \rightarrow \infty} g_1 g_2 \dots g_n \nu$$

*exists in the space  $P(M)$  of probability measures on  $M$ .*

*Proof.* Apply the martingale convergence theorem. □

**Proposition 4.5.** *Let  $\mu$  be a non-elementary probability measure on  $G$ , and let  $\nu$  be a  $\mu$ -stationary measure on  $\overline{X}^h$ . Then*

$$\nu(\overline{X}_F^h) = 0.$$

#### 4.2. End of proof of convergence.

**Proposition 4.6.** *For  $\mathbb{P}$ -a.e. sample path  $(w_n)$  there exists a subsequence  $(\rho_{w_n x_0})$  which converges to a horofunction in  $\overline{X}^h$ .*

*As a corollary,  $\mathbb{P}$ -a.e. sample path  $(w_n)$  there exists a subsequence  $(w_{n_k} x_0)$  which converges to a point in the Gromov boundary  $\partial X$ .*

**Proposition 4.7.** *Let  $\tilde{\nu}$  be a  $\mu$ -stationary measure on  $\partial X$ , and suppose that the sequence  $(w_n \tilde{\nu})$  converges to a  $\delta$ -measure  $\delta_\lambda$  on  $\partial X$ . Then  $(w_n x_0)$  converges to  $\lambda$  in  $X \cup \partial X$ .*

*Proof of Theorem 4.1.* Let  $\nu \in P(\overline{X}^h)$  a  $\mu$ -stationary measure, and denote  $\tilde{\nu} := \phi_*\nu \in P(\partial X)$ . By the martingale convergence theorem, for a.e.  $w_n$  we have  $(w_n)_*\nu \rightarrow \nu_w \in P(\overline{X}^h)$ . Then by pushing forward by  $\varphi_*$  one gets  $(w_n)_*(\tilde{\nu}) \rightarrow (\tilde{\nu})_w \in P(\partial X)$ . By  $\delta$ -hyperbolicity, if  $w_n x \rightarrow \xi \in \partial X$  then  $w_n \tilde{\nu} \rightarrow \delta_\xi$ . The sequence  $w_n x$  has at least one limit point  $\xi$  in  $\partial X$ , and for each limit point  $\xi$ ,  $w_{n_k} \tilde{\nu} \rightarrow \delta_\xi$ , but there can be only one limit point, as  $\lim_{n \rightarrow \infty} w_n \tilde{\nu}$  exists.  $\square$

## 5. APPENDIX

**5.1. Ergodic theorems.** In order to talk about asymptotic properties of random walks we need to have tools which assure us of the existence of various averages. Ergodic theorems provide such averages.

The most classical ergodic theorem is the *pointwise ergodic theorem* of Birkhoff.

**Definition 5.1.** A transformation  $T : (X, \mu) \rightarrow (X, \mu)$  of a measure space  $(X, \mu)$  is measure-preserving if  $\mu(A) = \mu(T^{-1}(A))$  for any measurable set  $A$ .

**Theorem 5.2** (Birkhoff). Let  $(X, \mu)$  be a measure space with  $\mu(X) = 1$ ,  $f : X \rightarrow \mathbb{R}$  be a measurable function, and  $T : X \rightarrow X$  a measure-preserving transformation. If  $f \in L^1(X, \mu)$ , then the limit

$$\bar{f}(x) := \lim_{n \rightarrow \infty} \frac{f(x) + f(T(x)) + \cdots + f(T^n(x))}{n}$$

exists for  $\mu$ -almost every  $x \in X$ .

We will derive Birkhoff's theorem from the more general *subadditive ergodic theorem* of Kingman.

A function  $a : \mathbb{N} \times X \rightarrow \mathbb{R}$  is a *subadditive cocycle* if

$$a(n+m, x) \leq a(n, x) + a(m, T^n x) \quad \text{for any } n, m \in \mathbb{N}, x \in X.$$

The cocycle is *integrable* if for any  $n$ , the function  $a(n, \cdot)$  belongs to  $L^1(X, \mu)$ . Assume moreover that

$$\inf \frac{1}{n} \int_X a(n, x) d\mu(x) > -\infty.$$

Then the following theorem holds.

**Theorem 5.3** (Kingman). Under the previous assumptions, there is an integrable, a.t.  $T$ -invariant function  $\bar{a}$  such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} a(n, x) = \bar{a}(x)$$

for almost every  $x \in X$ . Moreover, the convergence also takes place in  $L^1$ .

*Proof of Birkhoff's theorem.* We now see that Birkhoff's ergodic theorem follows as a corollary. In fact, if we let  $a(n, x) := \sum_{k=0}^{n-1} f(T^k x)$  then

$$a(n+m, x) = \sum_{k=0}^{n+m-1} f(T^k x) = a(n, x) + a(m, T^n x)$$

is actually an *additive cocycle*, thus it is subadditive.  $\square$

## 5.2. Conditional expectation.

**Theorem 5.4** (Radon-Nikodym). *Let  $(X, \mathcal{A}, \mu)$  be a probability space, and let  $\nu$  be a probability measure on  $\mathcal{A}$  which is absolutely continuous with respect to  $\mu$ . Then there exists a function  $f \in L^1(X, \mathcal{A}, \mu)$  such that*

$$\nu(A) = \int_A f \, d\mu.$$

Let us now consider a probability space  $(X, \mathcal{A}, \mu)$ , and  $\mathcal{B} \subset \mathcal{A}$  a smaller  $\sigma$ -algebra. Then the *conditional expectation* of a function  $f \in L^1(X, \mathcal{A}, \mu)$  with respect to  $\mathcal{B}$  is a function  $g \in L^1(X, \mathcal{B}, \mu)$  (in particular,  $g$  is  $\mathcal{B}$ -measurable) such that

$$\int_B f \, d\mu = \int_B g \, d\mu \quad \text{for all } B \in \mathcal{B}.$$

Usually one denotes such a  $g$  as  $\mathbb{E}(f \mid \mathcal{B})$ .

*Proof.* To prove the existence of conditional expectation, one considers the measure  $\nu$  on  $\mathcal{B}$  defined as

$$\nu(B) := \int_B f \, d\mu$$

Then, by the Radon-Nikodym theorem, the measure  $\nu$  is abs.cont. with respect to  $\mu$ , hence the Radon-Nikodym derivative  $g = \frac{d\nu}{d\mu}$  is a function in  $L^1(X, \mathcal{B}, \mu)$  which satisfies

$$\int_B f \, d\mu = \int_B g \, d\mu \quad \text{for all } B \in \mathcal{B}$$

as claimed. The uniqueness follows from the fact that two functions whose integrals agree on any set of the  $\sigma$ -algebra must agree almost everywhere (check this!).  $\square$

Given a set  $\mathcal{F}$  of functions, we denote as  $\sigma(\mathcal{F})$  the smallest  $\sigma$ -algebra for which all functions are measurable (i.e. the  $\sigma$ -algebra generated by all preimages of measurable sets) and denote

$$\mathbb{E}(f \mid \mathcal{F})$$

the conditional expectation of  $f$  with respect to  $\sigma(\mathcal{F})$ .

This has the intuitive interpretation of the expectation of  $f$  *once you know the values of the variables  $\mathcal{F}$* . Consider the coin toss  $(X_n) : \{0, 1\}^{\mathbb{N}} \rightarrow \{+1, -1\}$  where each  $X_n$  is i.i.d. and is  $+1$  with prob.  $1/2$ , and  $-1$  with prob.

1/2. Then the  $\sigma$ -algebra  $\sigma(X_1, \dots, X_n)$  is the set of functions on  $\Omega$  which *only depend* on the first  $n$  coordinates. Note that:

- (1) If  $f$  is independent of  $\mathcal{F}$ , then  $\mathbb{E}(f \mid \mathcal{F}) = \mathbb{E}(f)$ .
- (2) If  $f$  is  $\mathcal{F}$ -measurable, then  $\mathbb{E}(f \mid \mathcal{F}) = f$ .

Note that in particular if  $T : X \rightarrow X$  is a measure-preserving system, then one can define the  $\sigma$ -algebra  $\mathcal{F}_T$  of all  $T$ -invariant sets, and then the conditional expectation  $\mathbb{E}(f \mid \mathcal{F}_T) = \bar{f}$  is precisely the time average given by the ergodic theorem.

### 5.3. Martingales.

**Definition 5.5.** A sequence  $(X_n) : \Omega \rightarrow \mathbb{R}$  of measurable functions is a martingale if for any  $n$  we have

$$\mathbb{E}(X_{n+1} \mid X_1, \dots, X_n) = X_n.$$

A way to think of a martingale is that  $X_n$  is the payoff after  $n$  steps in a fair (i.e., zero-sum) game. That is, once you know the outcomes of the first  $n$  draws, the expected value of the payoff at step  $X_{n+1}$  is the previous payoff  $X_n$ .

In the example of the coin toss,  $Y_n := X_1 + \dots + X_n$  is a martingale. In fact

$$\mathbb{E}(Y_{n+1} \mid Y_1, \dots, Y_n) = \mathbb{E}(Y_n + X_{n+1} \mid Y_1, \dots, Y_n) = Y_n + \mathbb{E}(X_{n+1}) = Y_n.$$

**5.4. A bit of functional analysis.** Let  $M$  be a compact metric space. Then  $P(M)$  is the space of probability measures on  $M$ . We define convergence in the space of measure by saying that  $(\nu_n)$  converges to  $\nu$  in the *weak-\** topology if for any continuous  $f : M \rightarrow \mathbb{R}$ , we have

$$\int f d\nu_n \rightarrow \int f d\nu.$$

**Theorem 5.6** (Riesz-Markov-Kakutani). *The dual to the space  $C(M)$  of continuous functions on the compact metric space  $M$  is the space of signed Borel measures on  $M$ .*

**Theorem 5.7.** *The space  $P(M)$  is compact with respect to the weak-\** topology.

*Proof.* It is a closed subspace in the unit ball of the dual space of  $C(M)$ , in particular

$$P(M) := \{\varphi \in C(M)^* : \varphi \geq 0, \varphi(1) = 1\}$$

We say a functional is *positive* if  $\varphi(f) \geq 0$  whenever  $f$  is a non-negative function.  $\square$

**Theorem 5.8** (Alaoglu-Banach). *Let  $V$  be a normed vector space. Then the unit ball in its dual  $V^*$  is compact with respect to the weak-\** topology.

*Proof.* Recall that if  $\varphi \in V^*$  belongs to the unit ball, then  $|\varphi(v)| \leq \|v\|$  for any  $v \in V$ . Denote as  $B$  the unit ball in  $V$ , and  $B^*$  the unit ball in the dual, and consider the map  $F : B^* \rightarrow [-1, 1]^B$  defined as

$$F(\varphi) := (\phi(v))_{v \in B}$$

The map is injective as a functional is determined by its values on the unit ball. Moreover, by Tychonoff's theorem the cube  $[-1, 1]^B$  is compact as it is a product of compact spaces, and the image  $F(B^*)$  is closed in  $[-1, 1]^B$ , hence it is also compact.  $\square$

**5.5. Stationary measures.** A metric space  $M$  is called a  $G$ -space if there exists an action of  $G$  on  $M$  by homeomorphisms, i.e. a homomorphism  $\rho : G \rightarrow \text{Homeo}(M)$ .

**Lemma 5.9.** *Let  $M$  be a compact, metric  $G$ -space, and  $\mu$  a probability measure on  $G$ . Then there exists a  $\mu$ -stationary measure  $\nu$  on  $M$ .*

**Lemma 5.10.** *Let  $\nu$  be a  $\mu$ -stationary measure on a  $(G, \mu)$ -space  $M$ . Then for any  $f \in L^1(M, \nu)$ , the sequence*

$$X_n := \int_M f d(g_n \nu)$$

*is a martingale.*

**5.6. The Picard-Manin space.** If  $X$  is a smooth, projective, rational surface the group

$$N^1(X) := H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{R})$$

is called the *Néron-Severi group*. Its elements are Cartier divisors on  $X$  modulo numerical equivalence. The intersection form defines an integral quadratic form on  $N^1(X)$ . We denote  $N^1(X)_{\mathbb{R}} := N^1(X) \otimes \mathbb{R}$ .

If  $f : X \rightarrow Y$  is a birational morphism, then the pullback map  $f^* : N^1(Y) \rightarrow N^1(X)$  is injective and preserves the intersection form, so  $N^1(Y)_{\mathbb{R}}$  can be thought of as a subspace of  $N^1(X)_{\mathbb{R}}$ .

A *model* for  $\mathbb{C}\mathbb{P}^2$  is a smooth projective surface  $X$  with a birational morphism  $X \rightarrow \mathbb{C}\mathbb{P}^2$ . We say that a model  $\pi' : X' \rightarrow \mathbb{C}\mathbb{P}^2$  dominates the model  $\pi : X \rightarrow \mathbb{C}\mathbb{P}^2$  if the induced birational map  $\pi^{-1} \circ \pi' : X' \dashrightarrow X$  is a morphism. By considering the set  $\mathcal{B}_X$  of all models which dominate  $X$ , one defines the space of *finite Picard-Manin classes* as the inductive limit

$$\mathcal{Z}(X) := \lim_{X' \in \mathcal{B}_X} N^1(X')_{\mathbb{R}}.$$

In order to find a basis for  $\mathcal{Z}(X)$ , one defines an equivalence relation on the set of pairs  $(p, Y)$  where  $Y$  is a model of  $X$  and  $p$  a point in  $Y$ , as follows. One declares  $(p, Y) \sim (p', Y')$  if the induced birational map  $Y \dashrightarrow Y'$  maps  $p$

to  $p'$  and is an isomorphism in a neighbourhood of  $p$ . We denote the quotient space as  $\mathcal{V}_X$ . Finally, the *Picard-Manin space* of  $X$  is the  $L^2$ -completion

$$\mathcal{Z}(X) := \left\{ [D] + \sum_{p \in \mathcal{V}_X} a_p [E_p] : [D] \in N^1(X)_{\mathbb{R}}, a_p \in \mathbb{R}, \sum a_p^2 < +\infty \right\}.$$

In this paper, we will only focus on the case  $X = \mathbb{P}^2(\mathbb{C})$ . Then the Néron-Severi group of  $\mathbb{C}\mathbb{P}^2$  is generated by the class  $[H]$  of a line, with self-intersection  $+1$ . Thus, the Picard-Manin space is

$$\overline{\mathcal{Z}}(\mathbb{P}^2) := \left\{ a_0[H] + \sum_{p \in \mathcal{V}_{\mathbb{C}\mathbb{P}^2}} a_p [E_p], \sum_p a_p^2 < +\infty \right\}.$$

It is well-known that if one blows up a point in the plane, then the corresponding exceptional divisor has self-intersection  $-1$ , and intersection zero with divisors on the original surface.

Thus, the classes  $[E_p]$  have self-intersection  $-1$ , are mutually orthogonal, and are orthogonal to  $N^1(X)$ . Hence, the space  $\overline{\mathcal{Z}}(\mathbb{P}^2)$  is naturally equipped with a quadratic form of signature  $(1, \infty)$ , thus making it a Minkowski space of uncountably infinite dimension. Thus, just as classical hyperbolic space can be realized as one sheet of a hyperboloid inside a Minkowski space, inside the Picard-Manin space one defines

$$\mathbb{H}_{\mathbb{C}\mathbb{P}^2} := \{ [D] \in \overline{\mathcal{Z}} : [D]^2 = 1, [H] \cdot [D] > 0 \}$$

which is one sheet of a two-sheeted hyperboloid. The restriction of the quadratic intersection form to  $\mathbb{H}_{\mathbb{C}\mathbb{P}^2}$  defines a Riemannian metric of constant curvature  $-1$ , thus making  $\mathbb{H}_{\mathbb{C}\mathbb{P}^2}$  into an infinite-dimensional hyperbolic space. More precisely, the induced distance  $\text{dist}$  satisfies the formula

$$\cosh \text{dist}([D_1], [D_2]) = [D_1] \cdot [D_2].$$

Each birational map  $f$  acts on  $\overline{\mathcal{Z}}$  by orthogonal transformations. To define the action, recall that for any rational map  $f : \mathbb{C}\mathbb{P}^2 \dashrightarrow \mathbb{C}\mathbb{P}^2$  there exist a surface  $X$  and morphisms  $\pi, \sigma : X \rightarrow \mathbb{C}\mathbb{P}^2$  such that  $f = \sigma \circ \pi^{-1}$ . Then we define  $f^* = (\pi^*)^{-1} \circ \sigma^*$ , and  $f_* = (f^{-1})^*$ . Moreover,  $f_*$  preserves the intersection form, hence it acts as an isometry of  $\mathbb{H}_{\mathbb{P}^2}$ : in other words, the map  $f \mapsto f_*$  is a group homomorphism

$$\text{Bir } \mathbb{C}\mathbb{P}^2 \rightarrow \text{Isom}(\mathbb{H}_{\mathbb{P}^2})$$

hence one can apply to the Cremona group the theory of random walks on groups acting on non-proper  $\delta$ -hyperbolic spaces.

**Definition 5.11.** *The dynamical degree of a birational transformation  $f : X \dashrightarrow X$  is defined as*

$$\lambda(f) := \lim_{n \rightarrow \infty} \|(f^n)^*\|^{1/n}$$

where  $\|\cdot\|$  is an operator norm on the space of endomorphisms of  $H^*(X, \mathbb{R})$ .

Note that  $\lambda(f) = \lambda(gfg^{-1})$  is invariant by conjugacy. Moreover, if  $f$  is represented by three homogeneous polynomials of degree  $d$  without common factors, then the action of  $f^*$  on the class  $[H]$  of a line is  $f^*([H]) = d[H]$ , hence

$$\lambda(f) = \lim_{n \rightarrow \infty} \deg(f^n)^{1/n}.$$

Moreover, the degree is related to the displacement in the hyperbolic space  $\mathbb{H}_{\mathbb{P}^2}$ : in fact,

$$\deg(f) = f^*[H] \cdot [H] = [H] \cdot f_*[H] = \cosh d(x, fx)$$

if  $x = [H] \in \mathbb{H}_{\mathbb{P}^2}$ . As a consequence, the dynamical degree  $\lambda(f)$  of a transformation  $f$  is related to its translation length  $\tau(f)$  by the equation ([CL13], Remark 4.5):

$$\tau(f) = \lim_{n \rightarrow \infty} \frac{d(x, f^n x)}{n} = \lim_{n \rightarrow \infty} \frac{\cosh^{-1} \deg(f^n)}{n} = \log \lambda(f).$$

Hence, a Cremona transformation  $f$  is loxodromic if and only if  $\lambda(f) > 1$ .

## REFERENCES

- [MT] J. Maher, G. Tiozzo, *Random walks on weakly hyperbolic groups*, J. Reine Angew. Math. (2016).
- [MT2] J. Maher, G. Tiozzo, *Random walks, WPD actions, and the Cremona group*, preprint (2018), available at [arXiv:1807.10230\[math.GT\]](https://arxiv.org/abs/1807.10230).
- General references on random walks on groups**
- [BQ] Y. Benoist, J.-F. Quint, *Random walks on reductive groups*
- [BL] P. Bougerol, J. Lacroix, *Products of random matrices with applications to Schrödinger operators*
- [Fu] A. Furman, *Random walks on groups and random transformations*, in *Handbook of dynamical systems, Vol. I*
- [Wo] W. Woess, *Random walks on infinite graphs and groups*
- Boundary theory**
- [Fu1] H. Furstenberg, *Non-commuting random products*
- [Fu2] H. Furstenberg, *Poisson boundaries and envelopes of discrete groups*
- [Ka1] V. Kaimanovich, *Boundaries of invariant Markov operators: the identification problem*, in *Ergodic theory of  $\mathbb{Z}^d$  actions*, eds. M. Pollicott, K. Schmidt.
- [Ka2] V. Kaimanovich, *The Poisson formula for groups with hyperbolic properties*
- General references on hyperbolic geometry**
- [BH] M. Bridson, A. Haefliger, *Metric spaces of nonpositive curvature*
- [FM] B. Farb, D. Margalit, *A primer on the mapping class group*
- Applications to geometry and topology**
- [BF14] Mladen Bestvina and Mark Feighn, *Hyperbolicity of the complex of free factors*, Adv. Math. **256** (2014), 104-155.
- [CL13] Serge Cantat and Stéphane Lamy, *Normal subgroups in the Cremona group*, Acta Math. **210** (2013), no. 1, 31-94.
- [FM12] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012.
- [Hor16] Camille Horbez, *The Poisson boundary of  $\text{Out}(F_N)$* , Duke Math. J. **165** (2016), no. 2, 341-369.
- [KaM] V. Kaimanovich, H. Masur, *The Poisson boundary of the mapping class group*

- [KM] A. Karlsson, G. Margulis, *A Multiplicative Ergodic Theorem and Nonpositively Curved Spaces*
- [KK14] Sang-Hyun Kim and Thomas Koberda, *The geometry of the curve graph of a right-angled Artin group*, *Internat. J. Algebra Comput.* **24** (2014), no. 2, 121–169.
- [MM99] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, *Invent. Math.* **138** (1999), no. 1, 103–149.