Geodesic tracking for random walks on groups

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Summary

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2. Review on the mapping class group

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Question: What happens in the long run/on average?

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The sequence

$$x_0, w_1 x_0, w_2 x_0, \dots, w_n x_0, \dots$$

is called a sample path.

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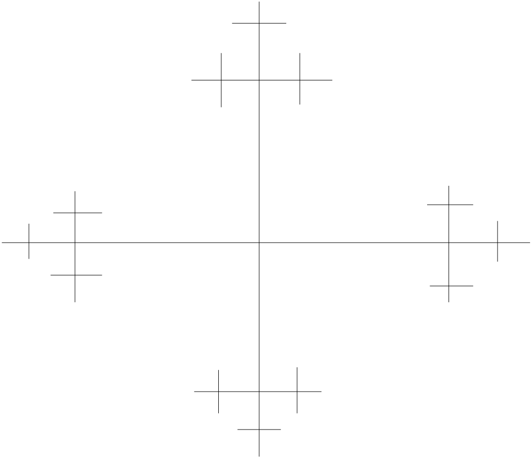

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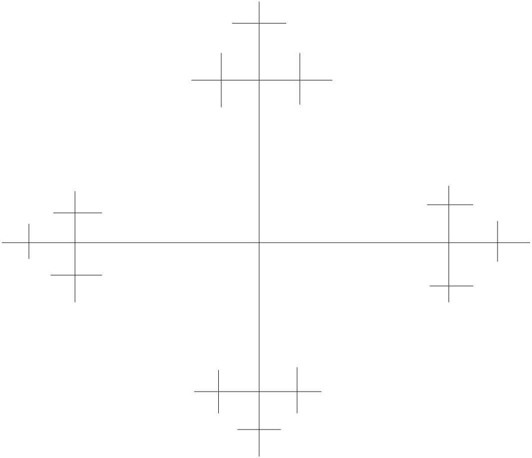

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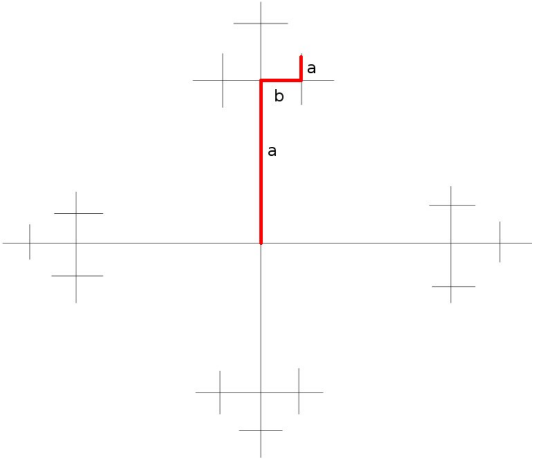

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   How about 3. ?

Sublinear tracking

We say that a random walk on G acting on the geodesic metric space (X, d) has the Sublinear Tracking property (ST) if for almost every sample path w_n there exists a geodesic ray $\gamma : [0, \infty) \to X$ such that

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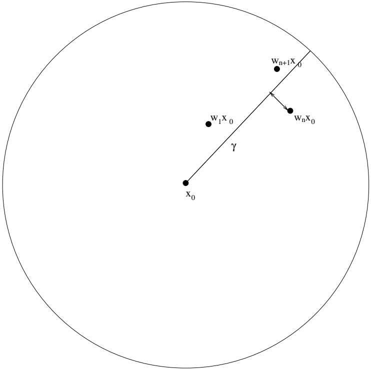

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2. Kaimanovich's criterion: sublinear tracking allows one to identify the Poisson boundary of the walk

$$H^\infty(G, \mu) = L^\infty(\partial X, \nu)$$

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- discrete groups of isometries of CAT(0) spaces [Karlsson-Margulis, ’99]

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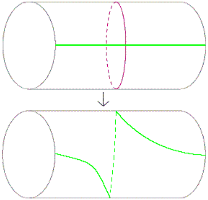
E.g.: Dehn twist around a curve



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[Duchin, '05] (ST) with restriction to thick part.

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Corollary

Poisson boundary = Thurston boundary [Kaimanovich-Masur]

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- Note: $\mathcal{T}(S)$ is NOT stably visible.

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- *relatively hyperbolic groups.*

The end

Thank you!