Shannon's theorem and Poisson boundaries for locally compact groups

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Summary

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joint with Behrang Forghani

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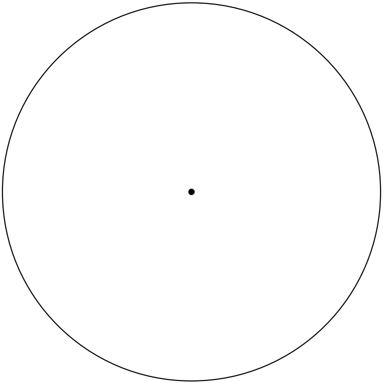

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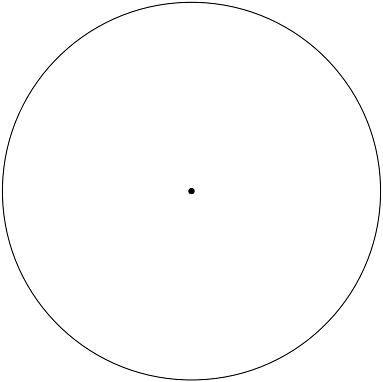

Theorem (Poisson representation)

There is a bijection

$$h^\infty(\mathbb{D}) \quad \leftrightarrow \quad L^\infty(S^1, \lambda)$$

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is harmonic on $\mathbb{D}$, i.e. $\Delta u = 0$. Here,

$$P_r(\theta) := \frac{1 - r^2}{1 + r^2 - 2r\cos\theta}$$

is the Poisson kernel.

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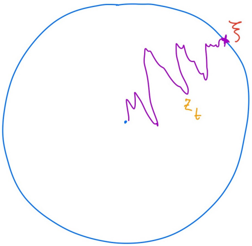

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Question. Can we generalize this to other groups $G \neq PSL_2(\mathbb{R})$?

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$$\nu = \int_G g\nu\, d\mu(g).$$

Random walks and μ-boundaries

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A space (B, ν) is a μ-boundary if there exists a measurable map

$$\text{bnd} : \Omega \to B$$

such that $\text{bnd} = \text{bnd} \circ T$.

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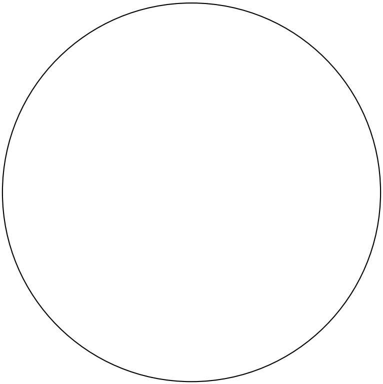

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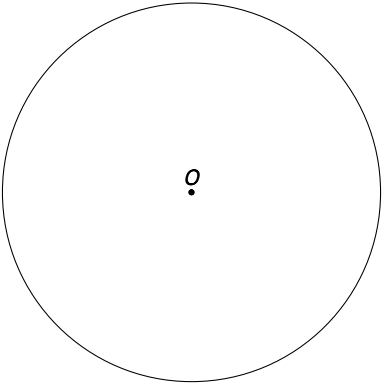

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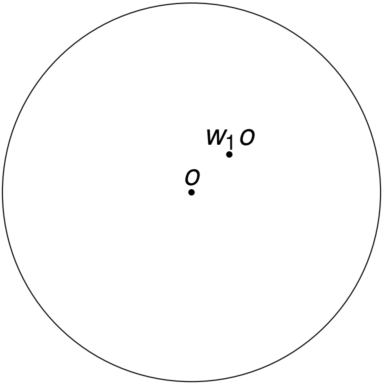

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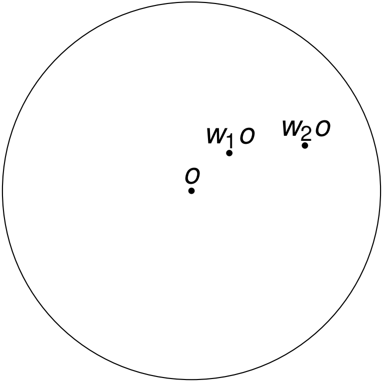

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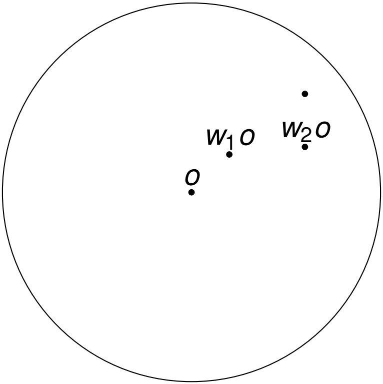

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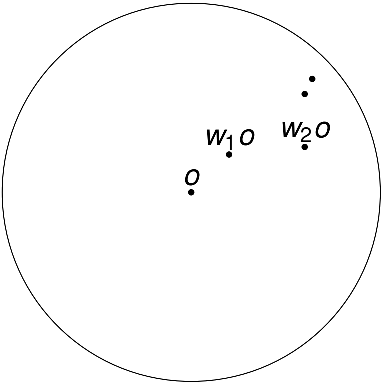

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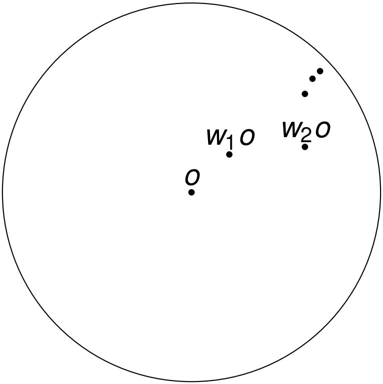

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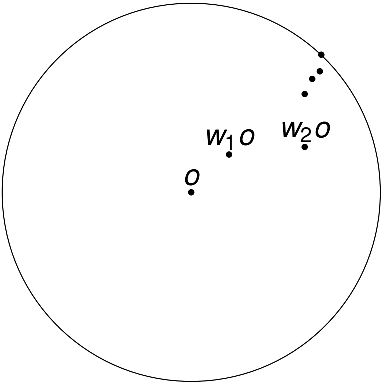

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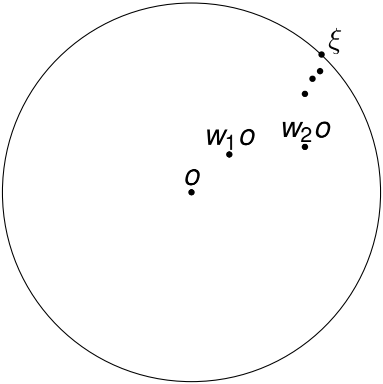

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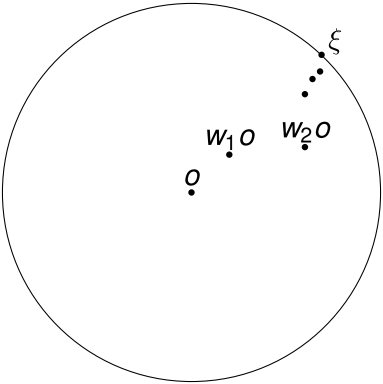

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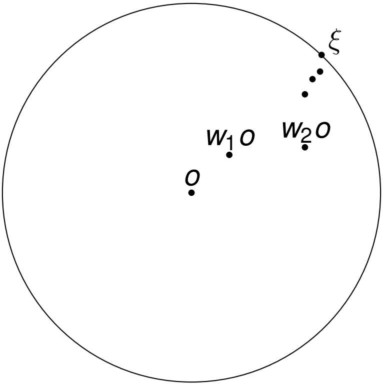

Moreover, $(\partial X, \nu)$ is a μ-boundary, given by the map

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Examples. Abelian groups; nilpotent groups

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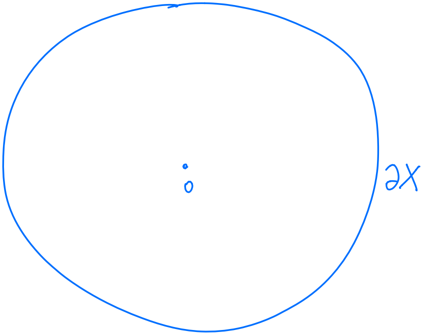

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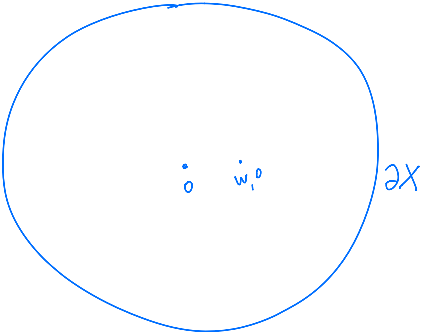

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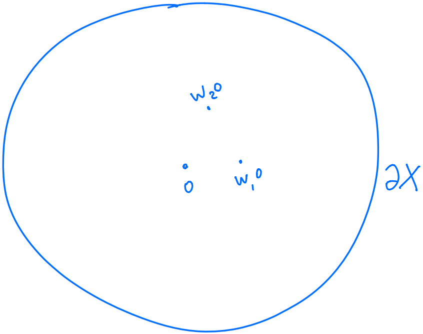

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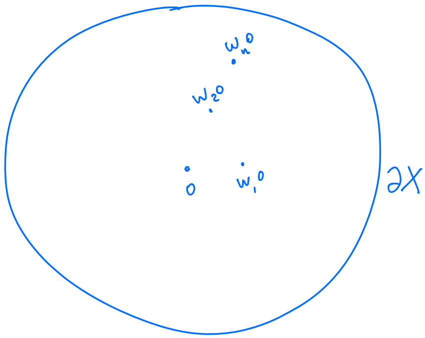

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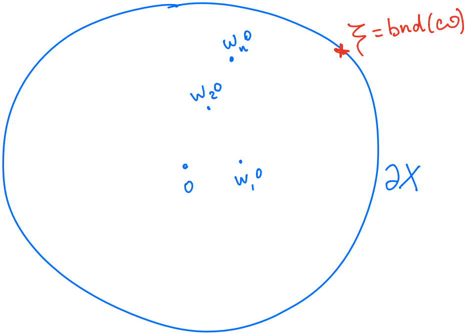

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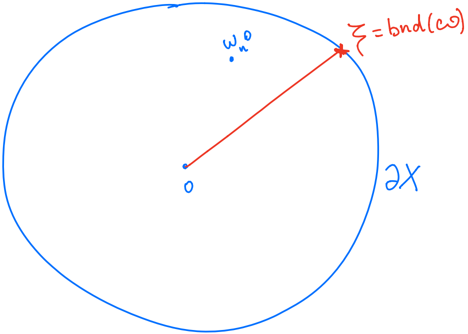

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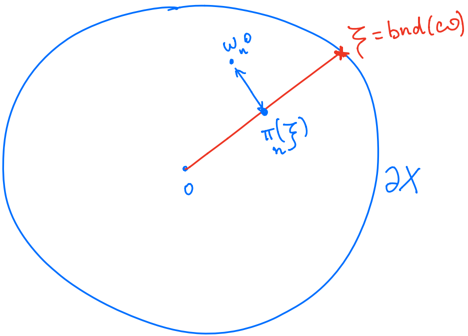

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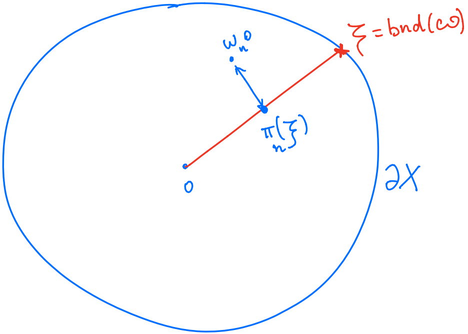

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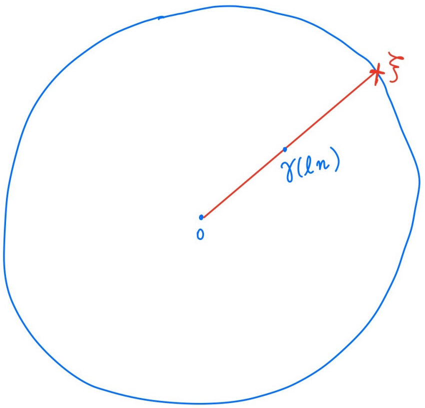

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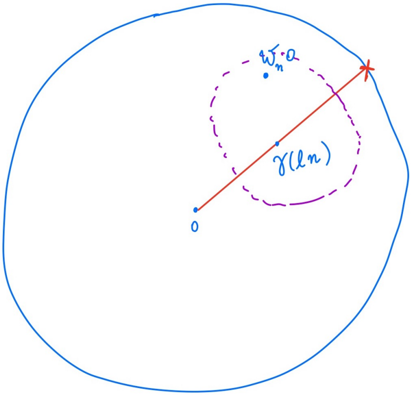

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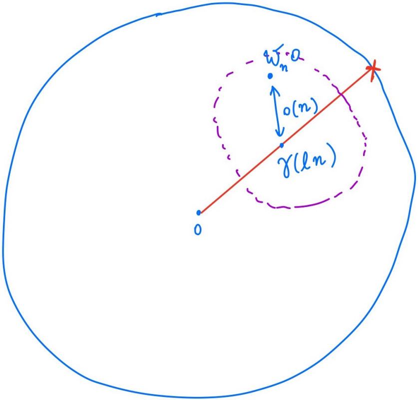

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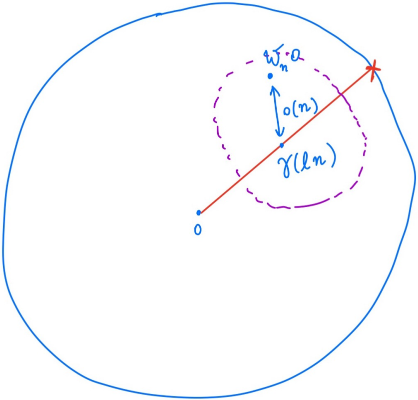

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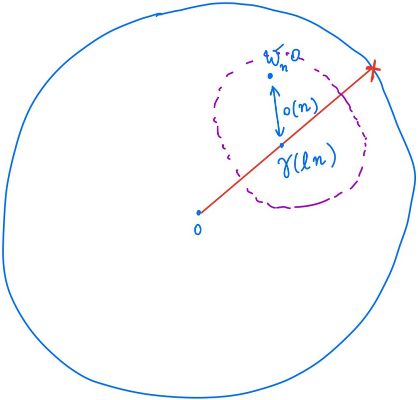

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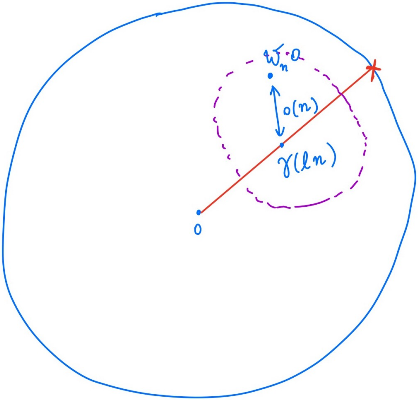

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(Kaimanovich-Woess '02) Ray criterion for invariant Markov operators (intermediate between discrete and lcsc groups)

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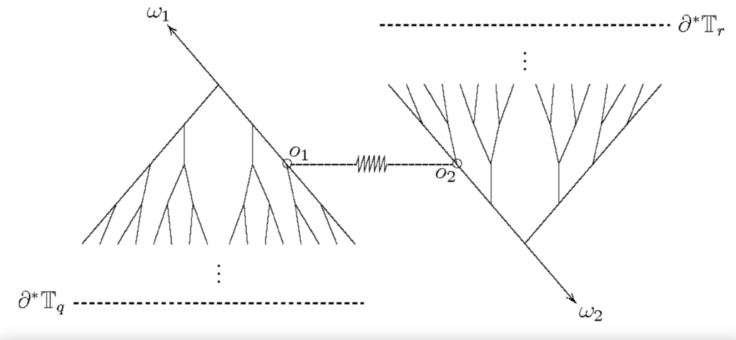

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Question. (Derriennic) Can we generalize to locally compact groups?

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Recall $h(\nu) =$ Furstenberg entropy.

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