Differentiable structures on the 7-sphere

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Introduction

The problem of classifying manifolds has always been central in geometry, and different categories of manifolds can a priori give different equivalence classes. The most famous classification problem of all times has probably been the Poincare' conjecture, which asserts that every manifold which is homotopy equivalent to a sphere is actually homeomorphic to it. A large family of invariants have been developed by algebraic topologists in the fifties to try to disprove the conjecture. In 1953, John Milnor analyzed some 7-manifolds arising as total spaces of fiber bundles and found examples of homotopy spheres which were not diffeomorphic to S^7. By looking at them more closely, he discovered that what we had found was not a counterexample to the conjecture, but still something remarkable. These manifolds were actually homeomorphic to the sphere, but they carried a non-standard differentiable structure: they are now known as *exotic spheres*. The aim of this paper is to present Milnor's construction: first, we will analyze the classification and topology of S^3 bundles over S^4; then we will develop the theory of characteristic casses, which will allow us to define and compute the invariant which distinguishes between different smooth structures; finally, we will use Morse theory to prove that these manifolds are homeomorphic to S^7.

1 Fiber bundles over spheres

The notion of fiber bundle is a generalization of that of *vector bundle*; intuitively, they are spaces that are locally products of two given spaces.

More formally, let X, Y, B be topological spaces and $p : X \to B$ be a continuous surjection, and G a group of homeomorphisms of Y. X is called the *total space*, B is the *base space*, and X is the *fiber*. The quintuple (X, B, Y, G, p) is called a *fiber bundle* if the following condition holds: there exists an open cover $\{U_i\}$ of B and homeomorphisms $\phi_i : p^{-1}(B) \to U_i \times Y$ (called *trivializations*) such that $p(\phi_i^{-1}(x, y)) = x$ for every $(x, y) \in U_i \times Y$, and for every pair of opens U_i, U_j with $U_i \cap U_j \neq \emptyset$ there exist continuous functions $g_{ij} : U_i \cap U_j \to G$ such that

$$\phi_i \circ \phi_j^{-1}|_{U_i \cap U_j} : U_i \cap U_j \times Y \to U_i \cap U_j \times Y$$

is of the form $(x, y) \mapsto (x, g_{ij}(x)y)$

A *morphism* of fiber bundles X, X' with same base and fiber is a continuous map $f : X \to X'$ which is expressible in trivializations as

$$\phi'_i \circ f \circ \phi_j^{-1} : U_j \cap U'_i \times Y \to U_j \cap U'_i \times Y$$

$$(x, y) \mapsto (x, \psi(y))$$

with $\psi(gy) = g\psi(y)$ for every $g \in G$.

If the fiber and base space are smooth manifolds, the structure group acts by diffeomorphisms, and all maps in the previous definition are *smooth*, then trivializations allow us to define a differentiable structure on the total space. This will be called a *smooth fiber bundle*.

A classical example of smooth fiber bundle is the *Hopf fibration* where $X = S^3$, $B = S^2$, $Y = S^1$, and p is the restriction of the defining map of $\mathbb{CP}^1$, when we identify X with the unit ball in $\mathbb{C}^2$ and B as $\mathbb{CP}^1$. Quaternionic analogues of this fibration will be our main objects of interest.

If the base space is the sphere S^n, there is an easy way to construct many different fiber bundles: given any map

$$f : S^{n-1} \to G$$

we can construct the total space

$$X_f := (U_0 \times Y) \sqcup (U_1 \times Y)/ \sim$$

where $(u, y) \sim (u, f(\pi(u))y)$, $U_0 = S^n \setminus \{\text{north pole}\}$, $U_1 = S^n \setminus \{\text{south pole}\}$, and $\pi : U_0 \cap U_1 \to S^{n-1}$ the projection onto the equator. The following theorem ([9], Th.18.5) identifies all equivalence classes of fiber bundles over a sphere:

Theorem 1.1. *For a fixed fiber Y, all fiber bundles over S^n are isomorphic to one obtained by the previous construction, and two such bundles are isomorphic iff the defining maps $S^{n-1} \to G$ are homotopic, i.e. they are classified by $\pi_{n-1}(G)$.*

We will now focus on the case $B = S^4$, $Y = S^3$ and $G = SO(4)$ the special orthogonal group (this group acts naturally on the unit ball in $\mathbb{R}^4$); by the preceding theorem, equivalence classes of such bundles are given by $\pi_3(SO(4))$.

To compute this group, let us consider S^3 as the unit ball in the 4-dimensional real vector space of quaternions, denoted by $\mathbb{H}$.

The map

$$SO(4) \to S^3 \times SO(3)$$

$$\phi \mapsto (\phi(1), \phi(1)^{-1}\phi)$$

is well defined, because since $SO(4)$ preserves the norm, then $\phi(1)$ belongs to the unit ball. On the other hand, $\phi(1)^{-1}\phi$ stabilizes 1, so it can be thought of as an element of $SO(3)$. It is quite easy to construct an inverse

$$S^3 \times SO(3) \to SO(4)$$

$$(u, \psi) \mapsto (v \mapsto u\psi(v))$$

where $SO(3)$ is identified with the subgroup of $SO(4)$ which acts trivially on the first coordinate, hence

$$SO(4) \cong S^3 \times SO(3)$$

It is also quite easy to see that $SO(3) \cong \mathbb{RP}^3$, via the map

$$\rho : S^3 \to SO(3)$$

$$u \mapsto (v \mapsto uvu^{-1})$$

The latter is an $\mathbb{R}$-linear, norm preserving map from $\mathbb{H}$ to itself, and $\rho(q)1 = 1$, which means we can view it as an element of $O(3)$. Since S^3 is connected, the image lands in the connected component of the identity, hence in $SO(3)$. To prove that the map is surjective, it is necessary to consider $\rho(e^{i\theta})$; it is clear that

$$\rho(e^{i\theta})1 = 1 \qquad \rho(e^{i\theta})i = i$$

while

$$\rho(e^{i\theta})j = e^{i\theta}je^{-i\theta} = e^{2i\theta}j = j\cos(2\theta) + k\sin(2\theta)$$

$$\rho(e^{i\theta})k = k\cos(2\theta) - j\sin(2\theta)$$

so all rotations in the (j, k) plane belong to the image; similarly, all rotations in the (i, j) and (i, k) plane do, and rotations w.r.t. the coordinate axes generate the group of rotations in $\mathbb{R}^3$. Finally, since ρ is also a group homomorphism whose kernel is $\{\pm 1\}$, the map is a 2-sheeted covering space of $S0(3)$, hence $\pi_3(SO(3)) \cong \pi_3(S^3) \cong \mathbb{Z}$.

This implies

$$\pi_3(SO(4)) \cong \pi_3(S^3) \oplus \pi_3(SO(3)) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Using the isomorphisms explicitly, we can describe completely representatives of all the equivalence classes of fiber bundles over S^4 with fiber S^3 and structure group $SO(4)$.

1. Since concatenation of maps with values in a topological group can be represented by pointwise multiplication, all elements of $\pi_3(S^3)$ can be represented by maps

$$\phi_a : S^3 \to S^3 \quad \forall a \in \mathbb{Z}$$

$$\phi_a(u) = u^a$$

2. Since ρ, being a finite cover, is an isomorphism on π_3, all elements of $\pi_3(SO(3))$ can be represented by

$$\rho \circ \phi_b : S^3 \to SO(3) \quad \forall b \in \mathbb{Z}$$

therefore for every $(a,b) \in \mathbb{Z}^2$ we have a map

$$\begin{array}{ccccc} S^3 & \to & S^3 \times SO(3) & \to & SO(4) \\ u & \mapsto & (u^a, \rho(u^b)) & \mapsto & (v \mapsto u^{a+b} v u^{-b}) \end{array}$$

We have proved

Proposition 1.2. *The map*

$$\mathbb{Z} \oplus \mathbb{Z} \to \pi_3(SO(4))$$

$$(h,j) \mapsto \phi_{hj}$$

with $\phi_{hj}(u) : v \mapsto u^h v u^j$ is a group isomorphism.

In the language of fiber bundles, a *real vector bundle* is a fiber bundle with fiber $\mathbb{R}^n$ and structure group contained in $GL(n,\mathbb{R})$. Given a fiber bundle with fiber S^{n-1} and structure group contained in $O(n)$, one can construct an associated vector bundle by considering the same transition functions as acting on all $\mathbb{R}^n$. Vicecersa, given a vector bundle with structure group contained in $O(n)$, by considering the unit ball in every fiber one gets a fiber bundle with fiber S^{n-1}.

We will denote by E_{hj} the vector bundle associated to the map ϕ_{hj}.

An alternative way to classify vector bundles is as pullbacks of the tautological vector bundle[1] over an appropriate Grassmannian: in the following, we will only need the

Proposition 1.3. *Every vector bundle over S^m of real fiber dimension n is isomorphic to the pullback of the tautological bundle on the Grassmannian of n-planes in $\mathbb{R}^{2n}$. Moreover, the map $\pi_{m-1}(SO(n)) \to \pi_m(G_n(\mathbb{R}^{2n}))$ is a group homomorphism.*

Proof. Let $\lambda : S^m \to [0,1]$ be a smooth function such that $\lambda \equiv 1$ in a neighbourhood of the north pole and $\lambda \equiv 0$ in a neighbourhood of the south pole. We can now define maps σ_0, σ_1 from the total space E given in terms of trivalizations

$$\begin{array}{cc} U_0 \times \mathbb{R}^n \to \mathbb{R}^n & U_1 \times \mathbb{R}^n \to \mathbb{R}^n \\ \sigma_0(u, v_0) = \lambda(u) v_0 & \sigma_1(u, v_1) = (1-\lambda(u)) v_1 \end{array}$$

such that

$$\sigma : E \to \mathbb{R}^n \oplus \mathbb{R}^n$$

$$\sigma(x) = (\sigma_0(x), \sigma_1(x))$$

is linear and injective on every fiber, hence the map

$$S^m \to G_n(\mathbb{R}^{2n})$$

[1] The tautological vector bundle over the Grassmannian of n-planes in $\mathbb{R}^m$ is the bundle whose fiber over the n-plane π is π itself.

$$x \mapsto \sigma(\text{fiber over } x)$$

identifies the bundle $E \to S^m$ with the pullback of the tautological bundle on the Grassmannian. In more explicit terms, the map $Ff : S^m \to G_n(\mathbb{R}^{2n})$ induced by f is

$$Ff(u) = \mathrm{Span}\{(\lambda(u)v_0, (1-\lambda(u))v_1), v_0, v_1 \in \mathbb{R}^n, v_1 = f(\pi(u))v_0 \text{ for } u \in U_0 \cap U_1\}$$

Given $f, g : S^{m-1} \to SO(n)$, a representative of the sum in $\pi_{m-1}(SO(n))$ is $(f \vee g) \circ p$, where $p : S^{m-1} \to S^{m-1} \vee S^{m-1}$ is the pinching map. We have the diagram

$$\begin{tikzcd}
S^{m-1} \arrow[r, "p"] \arrow[d] & S^{m-1} \vee S^{m-1} \arrow[r, "f \vee g"] \arrow[d] & SO(n) \\
S^m \arrow[r, "p"] & S^m \vee S^m \arrow[r, "Ff \vee Fg"] & G_n(\mathbb{R}^{2n})
\end{tikzcd}$$

where the lower row represents the sum $[Ff] + [Fg]$ in $\pi_m(G_n(\mathbb{R}^{2n}))$. Since one can check $F(f \vee g) = Ff \vee Fg$ by the explicit formula, it also represents $F([f] + [g])$.

$\square$

2 Characteristic classes

To distinguish between the smooth structures of total spaces of the sphere bundles we have considered so far, we need the machinery of characteristic classes.

Definition 2.1. *An* oriented vector bundle *is a real vector bundle with structure group contained in $SL(n, \mathbb{R})$.*

For every vector bundle E, let E_0 be the set of nonzero vectors in the total space, and E_x the fiber over $x \in B$. All cohomology groups are to be considered with $\mathbb{Z}$ coefficients.

If E is an oriented vector bundle, for every $x \in B$, a trivialization gives an isomorphism between E_x and $\mathbb{R}^n$, so you can pullback the standard orientation on $\mathbb{R}^n$, which is nothing else than a generator of $H^n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$, and get an element $u_x \in H^n(E_x, E_x \setminus \{0\})$. Since all elements in the structure group have positive determinants, this class does not depend on the trivialization.

Theorem 2.1. *(Thom isomorphism) Let E be an oriented vector bundle over the space B with fiber dimension n. There exists a unique cohomology class $u \in H^n(E, E_0)$ whose restriction to $H^n(E_x, E_x \setminus \{0\})$ is equal to u_x for every $x \in B$. The correspondence $y \mapsto y \cup u$ maps $H^j(E)$ isomorphically onto $H^{j+n}(E, E_0)$.*

The inclusion $E \to (E, E_0)$ gives rise in cohomology to a morphism $H^n(E, E_0) \to H^n(E)$, while the morphism $H^n(B) \to H^n(E)$ induced by the projection is an isomorphism, since the total space retracts to the zero section. The image of the class u given by the Thom isomorphism theorem in $H^n(B)$ via these two morphisms will be called the *Euler class* of the bundle and denoted by $e(E) \in H^n(B;\mathbb{Z})$.

By the long exact cohomology sequence of the pair (E, E_0) and using Thom isomorphism we get the *Gysin sequence*

$$\to H^i(B;\mathbb{Z}) \overset{\cup e}{\to} H^{i+n}(B;\mathbb{Z}) \overset{p_0^*}{\to} H^{i+n}(E_0;\mathbb{Z}) \to H^{i+1}(B;\mathbb{Z}) \overset{\cup e}{\to}$$

where $\cup e$ means taking cup product with the Euler class $e(E)$, while p_0^* is the morphism in cohomology induced by the restriction $p_0 : E_0 \to B$ of the projection defining the bundle.

Definition 2.2. *A* complex vector bundle *is a fiber bundle with fiber* $\mathbb{C}^n$ *and structure group contained in* $GL(n,\mathbb{C})$*.*

Let $E \to B$ be a complex vector bundle of complex fiber dimension n. The complex structure on the fiber induces a canonical orientation on the fiber itself, considered as a real vector space. Hence every complex vector bundle is also an oriented even-dimensional real vector bundle; let us define the n^{th} *Chern class* of E as the Euler class of the underlying real oriented vector bundle

$$c_n(E) := e(E) \in H^{2n}(B;\mathbb{Z})$$

In order to define the other Chern classes, let us endow every fiber with a hermitian metric (we can do it as long as the base space is paracompact); then, given a complex bundle $p : E \to B$ one can define the new bundle $E^\perp \to E_0$, where E_0 is the set of nonzero vectors in the total space. $E^\perp$ is defined as the subbundle of $p^*(E)$ whose fiber over every vector $v \in p^{-1}(x)$ is the $(n-1)$-dimensional space $v^\perp$ in $p^{-1}(x)$, orthogonal to v with respect to the chosen hermitian product.

Now, by the Gysin sequence we know that for $i < n$ the map $p_0^* : H^{2i}(B) \to H^{2i}(E_0)$ is an isomorphism. Hence we can define inductively the Chern classes $c_i(E)$ for $1 \leq i \leq n-1$ by

$$c_i(E) := (p_0^*)^{-1} c_i(E^\perp)$$

We define $c_i(E)$ to be zero for $i > n$.

We can define the *total Chern class*

$$c(E) = 1 + c_1(E) + c_2(E) + \cdots \in H^*(B;\mathbb{Z})$$

The properties of Chern classes are the following:

1. Naturality: If $E \to B$ is a vector bundle and $f : B' \to B$ a continuous map, $f^*E \to B'$ the pullback bundle, then $c_i(f^*E) = f^*c_i(E)$ for every i.
2. Triviality: If $\epsilon^k = B \times \mathbb{C}^k \to B$ is a trivial bundle, $c(E \oplus \epsilon^k) = c(E)$.
3. Product formula: If $E \oplus E'$ is the Whitney sum of E and E', then $c(E \oplus E') = c(E)c(E')$

Proof. 1. The naturality of the Euler class, and therefore of the top Chern class, follows from uniqueness in Thom isomorphism theorem; then, by induction: the map $f : B' \to B$ induces a map $f_0 : (f^*E)_0 \to E_0$ such that the bundle of orthogonal vectors $(f^*E)^\perp$ is precisely $f_0^*(E^\perp)$. Hence, considering the diagram

$$\begin{tikzcd} (f^*E)_0 \arrow[r, "f_0"] \arrow[d, "P_0"] & E_0 \arrow[d, "p_0"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

one has by inductive hypothesis $c_i((f^*E)^\perp) = f_0^* c_i(E^\perp)$, hence $c_i(f^*E) = (P_0^*)^{-1} c_i((f^*E)^\perp) = (P_0^*)^{-1} c_i(f_0^*(E^\perp)) = f^*(p_0^*)^{-1}(c_i(E^\perp)) = f^* c_i(E)$.

2. For $k = 1$, let us note that the bundle $E' = E \oplus \epsilon$ has a non-zero section $s : B \to (E \oplus \epsilon)_0$, which means $p_0 \circ s$ is the identity on B, hence on cohomology the composition

$$H^n(B) \xrightarrow{p^*} H^n(E') \to H^n(E'_0) \xrightarrow{s^*} H^n(B)$$

is the identity. By definition $p^*(e(E'))$ is the restriction of the Thom class u to E', and since the Thom class lies in $H^n(E', E'_0)$, its image in $H^n(E'_0)$ is zero. But the image of that class w.r.t. s^* is $e(E')$, hence $e(E \oplus \epsilon) = 0$, so $c_{n+1}(E \oplus \epsilon) = c_{n+1}(E)$. Moreover, one can check that $s^*((E \oplus \epsilon)^\perp) = E$, and by definition of Chern classes, $p_0^* c_i(E') = c_i((E')^\perp)$, hence $c_i(E') = s^* p_0^*(c_i(E)) = s^* c_i((E')^\perp) = c_i(E)$. The case $k > 1$ follows by induction.

3. See ([7], pag.164)

□

Given a complex vector bundle $\pi : E \to B$, we can define the *complex conjugate bundle* $\overline{E}$ as follows: if $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ are trivializations for E, the bundle $\overline{E}$ has total space E and trivializations

$$\pi^{-1}(U_i) \xrightarrow{\phi_i} U_i \times \mathbb{C}^n \xrightarrow{\tau} U_i \times \mathbb{C}^n$$

with $\tau(x, v) = (x, \overline{v})$ is complex conjugation in the $\mathbb{C}^n$-coordinate.

Lemma 2.2. *The Chern classes of the complex conjugate bundle $\overline{E}$ are given by*

$$c_i(\overline{E}) = (-1)^i c_i(E)$$

Proof. If $(v_1, \ldots, v_n)$ is a $\mathbb{C}$-basis of a fiber, the induced oriented $\mathbb{R}$-basis for E is $(v_1, iv_1, \ldots, v_n, iv_n)$, while for the conjugate bundle it is $(v_1, -iv_1, \ldots, v_n, -iv_n)$, hence the two induced orientations differ by $(-1)^n$, so $e(\overline{E}) = (-1)^n e(E)$. Since $\overline{E^\perp} \cong (\overline{E})^\perp$, then by induction $c_i(E) = (-1)^i c_i(\overline{E})$. □

Given a real vector bundle $E \to B$ with fiber $\mathbb{R}^n$, we can construct its *complexification* $E \otimes_{\mathbb{R}} \mathbb{C}$ as the complex vector bundle over B with fiber $\mathbb{C}^n$ and transition function given by the transition functions of E under the inclusion $GL(n, \mathbb{R}) \subseteq GL(n, \mathbb{C})$.

Definition 2.3. *For $i \geq 0$, let us define the i-th* Pontrjagin class *of the real vector bundle $E \to B$ as*

$$p_i(E) := (-1)^{2i} c_i(E \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(B; \mathbb{Z})$$

It follows from the definition that $p_i(E) = 0$ for $i > \frac{n}{2}$. The *total Pontrjagin class* is defined as

$$p(E) := 1 + p_1(E) + p_2(E) + \cdots \in H^*(B; \mathbb{Z})$$

.

The properties of Pontrjagin classes are quite similar to the properties of Chern classes, except for the product formula having a weaker form:

Proposition 2.3. *Let $E \to B$ be a real vector bundle.*

1. *For any continuous map $f : B' \to B$, $p_i(f^*E) = f^*(p_i(E))$ for every i.*
2. *If $\epsilon^k = B \times \mathbb{R}^k \to B$ is a trivial bundle, $p(E \oplus \epsilon^k) = p(E)$*
3. *If $F \to B$ is another vector bundle, $p(E \oplus F) - p(E)p(F)$ is a 2-torsion element (i.e. $2(p(E \oplus F) - p(E)p(F)) = 0$ in $H^*(B, \mathbb{Z})$)*

Proof. 1 and 2 follow directly from the corresponding properties of Chern classes. For 3, one has to notice that, since $E \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to $\overline{E \otimes_{\mathbb{R}} \mathbb{C}}$, by lemma 2.2 all odd Chern classes are 2-torsion elements, so one can write $c(E) = \sum_{i \equiv 0 \bmod 2} c_i(E) + \sum_{i \equiv 1 \bmod 2} \equiv \sum (-1)^i p_i(E)$ modulo 2-torsion elements, and then the result follows by the product formula for Chern classes. □

One more property we need is related to orientation:

Lemma 2.4. *If $E^- \to B$ is the bundle with opposite real orientation, then*

$$p_i(E^-) = p_i(E)$$

Proof. It is clear from the definition that reversing orientation of a real bundle reverses the Euler class. However, the real orientations induced on $E \otimes \mathbb{C}$ and $E^- \otimes \mathbb{C}$ are the same, hence they have the same Euler class and by induction the same Chern classes. □

Definition 2.4. *If M is a smooth manifold, we can define*

$$p_i(M) := p_i(TM)$$

Example 2.1. *Let us consider $S^n \subseteq \mathbb{R}^{n+1}$. The tangent and normal bundles to S^n are*

$$TS^n|_x = \{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\}$$

$$N|_x = \mathbb{R}x$$

hence the normal bundle N is trivial, and $TS^n \oplus N$ is also trivial. Therefore by property (2)

$$p(S^n) = p(TS^n) = p(TS^n \oplus N) = 1$$

For complex vector bundles, there is a direct relationship between Chern and Pontrjagin classes:

Lemma 2.5. *The complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ of the underlying real bundle of a complex vector bundle is isomorphic over $\mathbb{C}$ to the Whitney sum $E \oplus \overline{E}$.*

Proof. For every real vector space V, recall that the complexification $V \otimes \mathbb{C}$ is nothing else that the real vector space $V \oplus V$ with complex structure $J(x, y) = (-y, x)$. If V is the underlying real space of a complex vector space, we already have a multiplication by i on V. Now, the map

$$V \oplus V \to V \oplus V = V \otimes_{\mathbb{R}} \mathbb{C}$$

$$(x, y) \mapsto (x + y, -ix + iy)$$

is an isomorphism of real vector spaces which is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second one, i.e. $\phi(ix, 0) = J(\phi(x, 0))$ and $\phi(0, iy) = -J(\phi(0, y))$ hence it is a canonical $\mathbb{C}$-linear isomorphism $V \oplus \overline{V} \cong V \otimes_{\mathbb{R}} \mathbb{C}$. By defining this isomorphism on every fiber of a vector bundle you get the thesis. □

Lemma 2.6. *For any complex vector bundle E with fiber dimension n, the Chern classes determine the Pontrjagin classes by the formula*

$$1 - p_1 + p_2 - \dots = (1 - c_1 + c_2 - \dots)(1 + c_1 + c_2 + \dots)$$

Proof. $c(E \otimes_{\mathbb{R}} \mathbb{C}) = c(E)c(\overline{E}) = \sum_{i=0}^{\infty} c_i(E) \sum_{i=0}^{\infty} (-1)^i c_i(E)$. Moreover, if $k \equiv 1 \pmod 2$ then $c_k(E \otimes \mathbb{C}) = \sum_{0 \le i \le k} (-1)^i c_i(E) c_{k-i}(E) = 0$, so the total Chern class is just the sum of all even Chern classes. □

3 The signature of a manifold

Let M be a smooth compact oriented manifold of dimension $4n$ with fundamental class $\mu_M \in H^{4n}(M; \mathbb{R})$. Then we can define the bilinear form

$$H^{2n}(M; \mathbb{R}) \times H^{2n}(M; \mathbb{R}) \to \mathbb{R}$$

$$(x, y) \mapsto \langle \mu_M, x \cup y\rangle$$

which is symmetric. The signature of such a form is called the *signature* of M and denoted by $\sigma(M)$.

Hirzebruch's signature theorem relates the signature of a manifold to the evaluation of certain Pontrjagin classes; in order to state the theorem we need a little bit more machinery.

Consider the power series expansion

$$\frac{\sqrt{t}}{\tanh\sqrt{t}} = 1 + \frac{1}{3}t - \frac{1}{45}t^2 + \cdots + (-1)^{i-1}\frac{2^{2i}B_i}{(2i)!}t^i + ...$$

where B_i is the i^{th} Bernoulli number, and denote as λ_i the coefficient of t^i

A *partition* of a positive integer n is an unordered set $I = (i_1, \dots, i_k)$ of positive integers such that $i_1 + \cdots + i_k = n$. Let us fix n; for every partition I, let $\sum t^I$ be the smallest (in the sense of containing fewer monomials) symmetric polynomial in the variables $(t_1, \dots, t_n)$ which contains the monomial $t_1^{i_1} \dots t_k^{i_k}$; being a symmetric polynomial, it is a polynomial in the elementary symmetric functions, i.e. there exists a unique polynomial $s_I \in \mathbb{Z}[T_1, \dots, T_n]$ such that

$$s_I(\sigma_1, \dots, \sigma_n) = \sum t^I$$

where $\sigma_i \in \mathbb{Z}[t_1, \dots, t_n]$ is the i^{th} elementary symmetric polynomial in the variables $t_1, \dots, t_n$.

For every n, let us now define the polynomial $L_n \in \mathbb{Q}[T_1, \dots, T_n]$ by

$$L_n(T_1, \dots, T_n) = \sum_I \lambda_I s_I(T_1, \dots, T_n)$$

where the sum is over all partitions I of n, and $\lambda_I := \lambda_{i_1} \cdots \lambda_{i_k}$.

Example 3.1. *For $n = 1$, one has only the partition (1) and $\sum t^{(1)} = \sigma_1$, hence $L_1(T_1) = \lambda_1 T_1 = \frac{1}{3}T_1$.*
For $n = 2$, we have the partitions $(1,1)$ and (2). $\sum t^{(1,1)} = \sigma_2$ and $\sum t^{(2)} = \sigma_1^2 - 2\sigma_2$, therefore $L_2(T_1, T_2) = (\lambda_1)^2 T_2 + \lambda_2(T_1^2 - 2T_2) = \frac{1}{45}(7T_2 - T_1^2)$.

We are now ready to state the main theorem (for the proof see [7], chap.19):

Theorem 3.1. *Let M be a smooth, oriented, compact manifold of dimension $4n$ with fundamental class $[M] \in H^{4n}(M;\mathbb{Z})$. Then the signature is given by*

$$\sigma(M) = \langle [M], L_n(p_1(M), \dots, p_n(M)) \rangle$$

.

In the case of an 8-manifold we therefore have

$$\sigma(M) = \langle [M], \frac{1}{45}(7p_2(M) - p_1(M)^2) \rangle$$

4 The quaternionic projective space

Using quaternions, it is possible to construct an analogue of projective space of dimension $4n$. We will compute Pontrjagin classes of its tangent bundle.

Definition 4.1. *Let $n \geq 1$. The* quaternionic projective space $\mathbb{HP}^n$ *is defined as the quotient space*

$$\mathbb{H}^{n+1} \setminus \{0\}/\sim$$

where $(u_0, \ldots, u_n) \sim (\lambda u_0, \ldots, \lambda u_n)$ for every $\lambda \in \mathbb{H} \setminus \{0\}$.

We can construct a *tautological vector bundle* over $\mathbb{HP}^n$ by considering

$$\gamma_{\mathbb{H}}^n = \{(x, v) \in \mathbb{HP}^n \times \mathbb{H}^{n+1} \ s.t.\ [v] = x\}$$

The map $(x, v) \mapsto x$ gives a fiber bundle $\gamma_{\mathbb{H}}^n \to \mathbb{HP}^n$ which can be thought of both as a real vector bundle of fiber dimension 4 and a complex vector bundle of fiber dimension 2.

Proposition 4.1. *Let e be the Euler class of $\gamma_{\mathbb{H}}^n$. The cohomology ring of $\mathbb{HP}^n$ is*

$$H^*(\mathbb{HP}^n; \mathbb{Z}) = \frac{\mathbb{Z}[e]}{(e^{n+1})}$$

Proof. If E_0 is the set of nonzero vectors in the total space, the map

$$E_0 \to \mathbb{H}^{n+1}$$
$$(x, v) \mapsto \frac{v}{\|v\|}$$

is a homotopy equivalence between E_0 and the unit ball S^{4n+3} in $\mathbb{H}^{n+1}$ (the inverse is $S^{4n+3} \ni v \mapsto ([v], v) \in E_0 \subseteq \mathbb{HP}^n \times \mathbb{H}^{n+1}$), so the Gysin sequence of $\gamma_{\mathbb{H}}^n$ can be written as

$$\to H^{i+3}(S^{4n+3}; \mathbb{Z}) \to H^i(\mathbb{HP}^n; \mathbb{Z}) \xrightarrow{\cup e} H^{i+4}(\mathbb{HP}^n; \mathbb{Z}) \xrightarrow{\pi_0^*} H^{i+4}(S^{4n+3}; \mathbb{Z}) \to$$

By consiedring the cases $i = -4, -3, -2, -1$ one gets

$$H^0(\mathbb{HP}^n; \mathbb{Z}) = \mathbb{Z} \quad H^1(\mathbb{HP}^n; \mathbb{Z}) = H^2(\mathbb{HP}^n; \mathbb{Z}) = H^3(\mathbb{HP}^n; \mathbb{Z}) = 0$$

then, for $0 \leq i \leq 4m - 2$, multiplication by e gives an isomorphism

$$H^i(\mathbb{HP}^n; \mathbb{Z}) \cong H^{i+4}(\mathbb{HP}^n; \mathbb{Z})$$

hence by induction you get $H^i(\mathbb{HP}^n) = 0$ if $4 \nmid i$ while $H^{4i}(\mathbb{HP}^n)$ is generated by e^i for $0 \leq i \leq n$. (Higher classes are 0 since $\mathbb{HP}^n$ is a $4n$-manifold). $\square$

Corollary 4.2. *The total Pontrjagin class of $\gamma_{\mathbb{H}}^n$ is*

$$p(\gamma_{\mathbb{H}}^n) = 1 - 2e + e^2$$

Proof. Since the real fiber dimension is 4, $c_i(\gamma^n_{\mathbb{H}}) = 0$ for $i > 2$. Since $H^2(\mathbb{HP}^n;\mathbb{Z}) = 0$, $c_1(\gamma^n_{\mathbb{H}}) = 0$. Then the total Chern class has to be $c(\gamma^n_{\mathbb{H}}) = 1 + c_2(\gamma^n_{\mathbb{H}}) = 1 + e$. Therefore by lemma 2.6

$$1 - p_1(\gamma^n_{\mathbb{H}}) + p_2(\gamma^n_{\mathbb{H}}) = (1 + c_2(\gamma^n_{\mathbb{H}}))(1 + c_2(\gamma^n_{\mathbb{H}}))$$

hence the thesis. □

Let us recall that $\mathbb{HP}^1$ is diffeomorphic to S^4, via the map

$$\begin{array}{ccc} \mathbb{HP}^1 & \to & S^4 \subseteq \mathbb{R}^5 \\ (u_0, u_1) & \mapsto & \left(\frac{2\overline{u_1}u_0}{\|u_0\|^2+\|u_1\|^2}, \frac{\|u_0\|^2-\|u_1\|^2}{\|u_0\|^2+\|u_1\|^2} \right) \end{array}$$

Under this correspondence, $U_0 = \mathbb{HP}^1 \setminus \{u_0 = 0\}$ and $U_1 = \mathbb{HP}^1 \setminus \{u_1 = 0\}$ are charts for S^4.

Proposition 4.3. *The tautological bundle on $\mathbb{HP}^1$ is isomorphic, as oriented real vector bundle, to E_{01}.*

Proof. Let us compute transition functions for $\gamma = \gamma^1_{\mathbb{H}}$: a choice of trivalizations is

$$\gamma|_{\{u_0 \neq 0\}} \to \mathbb{H} \times \mathbb{H}$$

$$([1, t_1], (s_1, s_1 t_1)) \mapsto (s_1, t_1)$$

$$\gamma|_{\{u_1 \neq 0\}} \to \mathbb{H} \times \mathbb{H}$$

$$([t_2, 1], (s_2 t_2, s_2)) \mapsto (s_2, t_2)$$

hence the transition function is

$$g_{01} : U_0 \cap U_1 \to GL(\mathbb{H}) \subseteq SL(4, \mathbb{R})$$

$$[u_0, u_1] \mapsto (v \mapsto v u_0^{-1} u_1)$$

The transition function of E_{01} is $g'_{01}([u_0, u_1]) = \left(v \mapsto \frac{v u_0^{-1} u_1}{\|u_0^{-1} u_1\|}\right)$ hence we can get an isomorphism of real vector bundles by just rescaling every fiber: by defining $\lambda_0 : U_0 \to \mathbb{R}$, $\lambda_1 : U_1 \to \mathbb{R}$

$$\lambda_0([u_0, u_1]) = \begin{cases} 1 & \text{if } \|u_0\| \geq \|u_1\| \\ \|u_1 u_0^{-1}\| & \text{if } \|u_0\| < \|u_1\| \end{cases} \qquad \lambda_1([u_0, u_1]) = \begin{cases} 1 & \text{if } \|u_0\| \leq \|u_1\| \\ \|u_0 u_1^{-1}\| & \text{if } \|u_0\| > \|u_1\| \end{cases}$$

we can define the isomorphism on the two charts:

$$\begin{array}{c} \gamma|_{U_0} \cong U_0 \times \mathbb{H} \to U_0 \times \mathbb{H} \cong E_{01}|_{U_0} \\ (u, v) \mapsto (u, \lambda_0(u) v) \end{array} \qquad \begin{array}{c} \gamma|_{U_1} \cong U_1 \times \mathbb{H} \to U_1 \times \mathbb{H} \cong E_{01}|_{U_1} \\ (u, v) \mapsto (u, \lambda_1(u) v) \end{array}$$

□

Corollary 4.4.

$$p_1(E_{01}) = -2e$$

5 Milnor's λ invariant

In this section, we will use our knowledge of characteristic classes to "cook-up" an invariant which depends on the smooth structure and that will make us able to distiguish between some of the 7-manifolds we constructed so far.

This construction will apply to a smooth, closed, oriented 7-manifold M which is the boundary of an oriented 8-manifold and such that $H^3(M) = H^4(M) = 0$. Actually, Thom [10] proved that every oriented 7-manifold is the boundary of an 8-manifold, but we will not need this theorem in our arguments.

Let us fix an orientation $\mu \in H_7(M)$ and suppose $M = \partial B$, with B an oriented, smooth 8-manifold, and let us pick an orientation $\nu \in H_8(B, M)$ which induces the chosen orientation on M.

The relative cup product induces a symmetric biliner form

$$H^4(B, M; \mathbb{R}) \times H^4(B, M; \mathbb{R}) \to \mathbb{R}$$

$$(x, y) \mapsto \langle \nu, x \cup y \rangle$$

whose signature will be called the signature of B and denoted by $\sigma(B)$. By the vanishing of 3 and 4-dimensional cohomology, we have that the inclusion map

$$i : H^4(B, M) \to H^4(B)$$

is an isomorphism. The first Pontrjagin class $p_1(B)$ of the tangent bundle of B lies in $H^4(B)$; let

$$q(B) := \langle \nu, (i^{-1}p_1(B))^2 \rangle$$

Let us define the λ invariant of M (which a priori depends on B) as the residue class modulo 7

$$\lambda(M) := 2q(B) - \sigma(B) \pmod 7$$

Example 5.1. *For $M = S^7$ the standard unit sphere in $\mathbb{R}^8$, we can take the unit disk as B; since $H^4(B) = 0$, we have $\lambda(S^7) = 0$.*

Theorem 5.1. *The residue class of $\lambda(M)$ modulo 7 does not depend on the choice of B.*

Proof. Let B_1 and B_2 be two oriented manifolds with boundary M and orientations ν_1, ν_2 which induce μ on M. Then we can glue them and get a differentiable manifold $C = B_1 \cup B_2$ without boundary. Let us fix the orientation ν on C which induces ν_1 on B_1 and $-\nu_2$ on B_2. By Hirzebruch's signature theorem

$$\sigma(C) = \left\langle \nu, \frac{7p_2(C) - p_1(C)^2}{45} \right\rangle$$

which implies, by integrality of Pontrjagin classes,

$$2\langle \nu, p_1(C)^2 \rangle - \sigma(C) \equiv 0 \mod 7 \tag{1}$$

Now, for every n the relative Mayer-Vietoris sequence gives an isomorphism

$$h_* : H_n(B_1, M) \oplus H_n(B_2, M) \to H_n(C, M)$$

and similarly a corresponding one h^* in cohomology. Moreover, from the long exact sequence of the pair (C, M) you get a morphism $j_* : H_n(C) \to H_n(C, M)$ and a dual one j^* in cohomology. From the hypotheses on M, we know j^* is an isomorphism for $n = 4$. Hence, every $\alpha \in H^4(C)$ is of the form $\alpha = j^*(h^*)^{-1}(\alpha_1, \alpha_2)$ with $\alpha_i \in H_4(B_i, M)$. Therefore

$$\langle \nu, \alpha^2 \rangle = \langle \nu, j^*(h^*)^{-1}(\alpha_1^2, \alpha_2^2) \rangle = \langle j_*\nu, (h^*)^{-1}(\alpha_1^2, \alpha_2^2) \rangle = \langle \nu_1, \alpha_1^2 \rangle - \langle \nu_2, \alpha_2^2 \rangle$$

hence

$$\sigma(C) = \sigma(B_1) - \sigma(B_2) \tag{2}$$

Now, the inclusion $\iota : B_1 \to C$ is an embdding, so $TB_1 = \iota^*(TC)$ and $p_1(B_1) = \iota^* p_1(C)$. Similarly for B_2, so the restriction map $H^4(C) \to H^4(B_1) \oplus H^4(B_2)$ sends $p_1(C)$ to $(p_1(B_1), p_1(B_2))$. If we denote by i_l the isomorphism $i_l : H^4(B_l, M) \to H^4(B_l)$, the same computation as before with $\alpha = p_1(C)$, $\alpha_1 = i_1^{-1} p_1(B_1)$, $\alpha_2 = i_2^{-1} p_1(B_2)$ shows

$$\langle \nu, p_1^2(C) \rangle = \langle \nu_1, (i_1^{-1} p_1(B_1))^2 \rangle - \langle \nu_2, (i_2^{-1} p_1(B_2))^2 \rangle = q(B_1) - q(B_2)$$

which together with (2) and (1) proves the theorem. □

Notice that, if $f : M_1 \to M_2$ is an orientation-preserving diffeomorphism, the previous proof shows $\lambda(M_1) = \lambda(M_2)$.

6 Computing λ

For any $(h, j) \in \mathbb{Z}^2$ we can now compute the characteristic classes of the vector bundle E_{hj}; let ι be the generator of $H^4(S^4)$ corresponding to the Euler class e of the tautological bundle on $\mathbb{HP}^1$ under the identification $S^4 \cong \mathbb{HP}^1$.

Proposition 6.1.

$$p_1(E_{hj}) = 2(h - j)\iota$$
$$e(E_{hj}) = (h + j)\iota$$

Proof. The computation of the Pontrjagin class is achieved in 3 steps:

1. $p_1(E_{hj})$ is linear in h and j. This is because the map which assigns to (h, j) the class $p_1(E_{hj}) \in H^4(S^4)$ is the composition of three group homomorphisms: first, the isomorphism $\mathbb{Z}^2 \to \pi_3(SO(4))$ described in proposition 1.2, then the homomorphism $\pi_3(SO(4)) \to \pi_4(G_4(\mathbb{R}^8))$ given by proposition 1.3; the third map is

$$\pi_4(G_4(\mathbb{R}^8)) \to H^4(S^4)$$
$$[f] \mapsto p_1(f^*(\gamma^4))$$

where γ^4 is the tautological 4-plane bundle on $G_4(\mathbb{R}^8)$. This is also a group homomorphism, because $p_1(f^*(\gamma^4)) = f^*(p_1(\gamma^4))$ and the map $\pi_n(X) \to \mathrm{Hom}(H^k(X), H^k(S^n))$ given by $f \to f^*$ is a group homomorphism.

2. $p_1(E_{hj}) = c(h-j)\iota$ for some constant $c \in \mathbb{Z}$. Let us consider the bundle $\overline{E}_{hj}$ given by taking the quaternionic conjugate of every fiber. Since it reverses the orientation of fibers, $p_1(E_{hj}) = p_1(\overline{E}_{hj})$ (lemma 2.4). On the other hand, by conjugating the transition function $u \mapsto \frac{u^h v u^j}{\|u\|^{h+j}}$ one gets the transition function of the bundle $E_{-j,-h}$, hence $p_1(E_{hj}) = p_1(E_{-j,-h})$ for every $j, h \in \mathbb{Z}$.

3. $p_1(E_{hj}) = 2(h-j)\iota$. This follows from corollary 4.4 setting $h = 0, j = 1$.

Similarly, the Euler class is linear in h and j, and, since reversing orientation carries such class to the negatve of itself, $e(E_{hj}) = c(h+j)\iota$ for some $c \in \mathbb{Z}$. Again, by comparison with the Euler class of the tautological bundle, we get $c = 1$. □

Let us now consider, for every (h, j), the sphere bundle obtained by taking the unit sphere in every fiber of E_{hj}; the total space M_{hj} is a smooth 7-manifold which can be defined as

$$(\mathbb{H} \times S^3) \sqcup (\mathbb{H} \times S^3)/\sim$$

$$(u, v) \sim \left(u^{-1}, \frac{u^h v u^j}{\|u\|^{h+j}}\right) \text{ for } u \in \mathbb{H} \setminus \{0\}, v \in S^3 \subseteq \mathbb{H}$$

The cohomology of M_{hj} can be computed via the Gysin sequence of E_{hj}, since M_{hj} is homotopy equivalent to the set of nonzero vectors in such bundle.

One can easily see $H^r(M_{hj}) = 0$ for every $r \neq 0, 3, 4, 7$, and $H^0(M_{hj}) = H^7(M_{hj}) = \mathbb{Z}$. The computation of H^3 and H^4 is a little bit more interesting:

$$0 \to H^3(M) \to H^0(S^4) \xrightarrow{\cup e} H^4(S^4) \to H^4(M) \to 0$$

therefore since $e(E_{hj}) = (h+j)\iota$

$$H^3(M_{hj}) \cong \begin{cases} 0 & \text{if } h+j \neq 0 \\ \mathbb{Z} & \text{if } h+j = 0 \end{cases} \qquad H^4(M) \cong \frac{\mathbb{Z}}{(h+j)\mathbb{Z}}$$

It is then clear that if $h + j = 1$, M_{hj} has the *same cohomology* as the 7-sphere. By Poincare' duality, it also has the same homology. Now, from the exact sequence of the fibration $M_{hj} \to S^4$ one gets $\pi_1(M_{hj}) = 0$; then by Hurewicz theorem $\pi_7(M_{hj}) = H_7(M_{hj}) = \mathbb{Z}$, hence we have a map $S^7 \mapsto M_{hj}$ which induces an isomorphism on π_7 and H_7. It also trivially induces an isomorphism on H_0, and since all other homology groups are 0, it induces an isomorphism on all homology groups. Now, *a continuous map between connected, simply connected CW complexes which induces an isomorphism in all homotopy groups is a homotopy equivalence*[[1], Prop.4.74], hence we have proved

Proposition 6.2. *For $h + j = 1$, the manifold M_{hj} is homotopy equivalent to S^7.*

This gives us the motivation to restrict our attention to the case $h + j = 1$. Let k be an odd integer and let h and j such that $h - j = k$, $h + j = 1$. The total space of the sphere bundle M_{hj} will be from now on denoted as M_k.

Such manifolds certainly have $H^3(M_k) = H^4(M_k) = 0$; moreover, M_k is the boundary of the 8-manifold B_k, the total space of the unit disk bundle in E_{hj}; therefore we can compute $\lambda(M_k)$.

Theorem 6.3.

$$\lambda(M_k) \equiv k^2 - 1 \pmod 7$$

Proof. Since the inclusion of the zero section in B_k is a homotopy equivalence, it induces an isomorphism in cohomology $H^*(S^4) \to H^*(B_k)$, so $H^4(B_k)$ is cyclic and $\sigma(B_k)$ can only be ± 1. Let us fix the orientation on B_k (and therefore also on M_k) so that $\sigma(B_k) = 1$. Now, the total space B_k embeds into the total space of the vector bundle E_{hj} (with $h = \frac{1+k}{2}$, $j = \frac{1-k}{2}$), hence to compute characteristic classes of TB_k it is sufficient the study the tangent bundle TE_{hj}. By choosing a Riemannian metric on each fiber, we determine an isomorphism

$$TE_{hj} \cong p^*(TS^4) \oplus p^*(E_{hj})$$

hence by proposition 2.3 (there are no torsion elements in $H^4(E_{hj}) \cong H^4(S^4)$)

$$p_1(TE_{hj}) = p^*(p_1(TS^4)) + p^*(p_1(E_{hj})) = p^*(2k\iota)$$

where we used proposition 6.1 and example 2.1. Hence

$$p_1(B_k) = 2k\alpha$$

where α is the pullback of ι via the map $B_k \to E_{hj} \to S^4$. Now

$$\lambda(M_k) = 2q(B_k) - \sigma(B_k) = 2\langle \nu, (i^{-1}(2k\alpha))^2 \rangle - 1 \equiv k^2 - 1 \pmod 7$$

□

Since we already computed λ for the standard smooth structure on S^7, we have proved

For $k^2 \not\equiv 1$ mod 7, M_k is a homotopy 7-sphere not diffeomorphic to S^7.

At this point, Milnor thought he had found a counterexample to the Generalized Poincare' Conjecture. Actually, not quite...

7 A (little) bit of Morse theory

It turns out that these manifolds M_k are actually homeomorphic to S^7, so they are examples of exotic differentiable structures on the 7-sphere.

In order to prove that, we need some basic results from Morse theory. Let M be a smooth, compact manifold, and $f : M \to \mathbb{R}$ a smooth function; f is called a *Morse function* if every critical point of f is non-degenerate, i.e. if at any point where the differential of f vanishes, the hessian matrix $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is non-singular. The goal of Morse theory is to infer the topology of M by studying the behaviour of f at the critical points.

A fundamental lemma one needs to do so is

Lemma 7.1. *If x_0 is a critical point of f, there exists coordinates $v_1, \ldots, v_n$ on a neighbourhood V of x_0 such that $f|_V$ is expressible as*

$$f(v_1, \ldots, v_n) = f(x_0) + v_1^2 + \cdots + v_k^2 - v_{k+1}^2 - \cdots - v_n^2$$

Proof. We can of course assume that we are working in a convex neighbourhood of 0 in $\mathbb{R}^n$, and $f(0) = 0$. Then

$$f(x_1, \ldots, x_n) = \int_0^1 \frac{df(tx_1, \ldots, tx_n)}{dt} dt = \int_0^1 \sum_{1=1}^{n} \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) x_i dt$$

Therefore the functions

$$g_i(x_1, \ldots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \ldots, tx_n) dt$$

are smooth functions such that

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n)$$

and $g_i(0) = \frac{\partial f}{\partial x_i}(0)$. Applying the same trick to the g_i, we get

$$f(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \ldots, x_n)$$

for some smooth h_{ij}. Then by taking $\tilde{h}_{ij} = \frac{h_{ij} + h_{ji}}{2}$ we can assume the matrix $(\tilde{h}_{ij}(0))$ to be symmetric and, by the non-degeneracy condition, non-singular. The rest of the proof is by induction: suppose that there exists coordinates $u_1, \ldots, u_n$ in a neighbourhood of 0 so that

$$f = \pm u_1^2 \pm \cdots \pm u_{r-1}^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \ldots, u_n)$$

for H_{ij} smooth functions with $H_{ij} = H_{ji}$. Being the matrix $H_{ij}(0)$ equal up to sign to the hessian of f at 0, we can perform a linear change in the last $n-r+1$ coordinates so that $H_{rr}(0) \neq 0$. The function $G(u_1, \ldots, u_n) = \sqrt{|H_{rr}(u_1, \ldots, u_n)|}$

is smooth in a (possibly smaller) neighbourhood of 0, and we can introduce the new variables $v_1, \dots, v_n$:

$$\begin{cases} v_i = u_1 \quad \text{for } i \neq r \\ v_r(u_1, \dots, u_n) = G(u_1, \dots, u_n) \left(u_r + \sum_{i=r+1}^n u_i \frac{H_{ir}(u_1,\dots,u_n)}{|H_{rr}(u_1,\dots,u_n)|} \right) \end{cases}$$

By the inverse function theorem, the v_i are coordinate functions in a neighbourhood of 0; f can be expressed there as

$$f = \sum_{i \leq r} \pm v_i^2 + \sum_{i,j \geq r+1} v_i v_j H'_{ij}(v_1, \dots, v_n)$$

which completes the inductive step. □

In our case, only a simple instance of the main theorem of Morse theory is required, namely

Theorem 7.2. *Let M be a closed, smooth manifold of dimension n. If there exists a Morse function $f : M \to \mathbb{R}$ with only two critical points, then there exists a homeomorphism of M onto S^n which is a diffeomorphism except possibly at one point.*

Proof. By compactness, f has at least a minimum x_0 and a maximum x_1; since there are only two critical points, they have to be exactly x_0 and x_1. By acting with a linear transformation, we can reduce to the case $f(x_0) = 0$, $f(x_1) = 1$. By the previous lemma, there exists V a neighbourhood of x_0 and coordinates $v_1, \dots, v_n$ on V such that $f(v_1, \dots, v_n) = v_1^2 + \dots v_n^2$ (there are no negative terms by the minimum condition). We can now define a Riemannian metric on V of the form $ds^2 = dv_1^2 + \dots + dv_n^2$; by using partitions of unity, one can extend that to a Riemannian metric on M. This allows us to define the gradient vector field ∇f, which is singular exactly at x_0 and x_1. The differential equation

$$\frac{dx}{dt} = \frac{\nabla f}{\|\nabla f\|^2} \tag{3}$$

can be solved explicitly on $V \setminus \{x_0\}$: for any n-tuple $\underline{a} = (a_1, \dots, a_n)$ such that $a_1^2 + \dots a_n^2 = 1$, the trajectory

$$x_{\underline{a}}(t) = (a_1 t^{1/2}, \dots, a_n t^{1/2})$$

is an integral curve for $t \in (0, \epsilon)$, and can be extended uniquely as long as it does not land on x_0 or x_1. Since $x_{\underline{a}} \to 0$ as $t \to 0$, one can also extend it continuously (though not smoothly) to $t = 0$.

Let us note that, if $x(t)$ satisfies (3),

$$\frac{\partial f(x(t))}{\partial t} = f_* \left(\left. \frac{\nabla f}{\|\nabla f\|^2} \right|_{x(t)} \right) = \langle \nabla f, \frac{\nabla f}{\|\nabla f\|^2} \rangle = 1$$

hence if $x(0) = 0$, $f(x(t)) = t$; this tells us that all these solutions can be extended smoothly to $x_{\underline{a}} : (0,1) \to M \setminus \{x_0, x_1\}$; by letting $t \to 1$ one notices they can also be extended continuously to $[0,1]$. Now, the map Φ defined on the closed unit disk $D^n \subseteq \mathbb{R}^n$

$$\Phi : D^n \to M$$

$$(x_1, \dots, x_n) \mapsto x_{(\frac{x_1}{\|x\|}, \dots, \frac{x_n}{\|x\|})}(\|x\|^2)$$

is certainly smooth on the interior of D^n (since the flow of a nonsingular smooth vector field depends smoothly on the initial condition) except possibly at 0; by expressing it in the coordinates $(v_1, \dots, v_n)$ in a neighbourhood of 0, one realizes

$$\Phi(x_1, \dots, x_n) = (a_1 t^{1/2}, \dots, a_n t^{1/2}) = (x_1, \dots, x_n)$$

because $a_i = \frac{x_i}{\|x\|}$, $t = \|x\|^2$, hence it is clearly a local diffeomorphism. Now, we claim that the restriction of Φ to the interior of D^n is a diffeomorphism onto $M \setminus \{x_1\}$: it is injective by the uniqueness of integral curves of smooth vector fields on compact manifolds, and it is surjective because by reversing the flow one can get arbitrarily close to x_0; moreover, the backward flow gives us an explicit smooth inverse. Now, Φ maps the boundary of D^n to $\{x_1\}$, hence it induces a continuous bijective map $S^n \to M$; since S^n is compact and M is Hausdorff, this map is a homeomorphism. $\square$

In the end, we can apply this theorem to our manifolds M_k. Recall M_k is given as $(\mathbb{H} \times S^3) \sqcup (\mathbb{H} \times S^3)/\sim$ with $(u, v) \sim (u', v') = (u^{-1}, \frac{u^h v u^{1-h}}{\|u\|})$, $h = \frac{1+k}{2}$. Let us consider the function $f : M_k \to \mathbb{R}$ given in the first chart by

$$f(u, v) = \frac{\Re v}{(1 + \|u\|^2)^{1/2}} \tag{4}$$

where $\Re v$ stands for the real part of the quaternion v. This map has on this chart only two critical points, namely for $u = 0, v = \pm 1$, and they are non-degenrate; on the overlap of the two charts we get the expression

$$f(u', v') = \frac{\Re((u')^h v' (u')^{1-h})}{(1 + \|u'\|^2)^{1/2}}$$

and by setting the new coordinates (u'', v') (instead of (u', v')) with $u'' = v'u'$, we have

$$f(u'', v') = \frac{\Re(u'')}{(1 + \|u''\|^2)^{1/2}}$$

This means that f can be extended smoothly to the second chart, giving a smooth function on all M_k, and moreover it is immediate to check that f has no critical points on the second chart; therefore we are in the hypothesis of the theorem, and finally we get

Proposition 7.3. *For every $k \in \mathbb{Z}$, M_k is homeomorphic to S^7.*

8 How many?

Since there are 4 possible values of $k^2 - 1$ with $k \in \mathbb{Z}/7\mathbb{Z}$, so far we have constructed 4 different smooth structures on the 7-sphere. It is quite easy to get 7 of them, by taking *connected sums.*

Definition 8.1. *Let M_1, M_2 be two oriented smooth manifolds of same dimension n, and fix $h_1 : \mathbb{R}^n \to M_1$, $h_2 : \mathbb{R}^n \to M_2$ two imbeddings, such that h_1 preserve orientation and h_2 reverse it. Then you can define an oriented smooth manifold*

$$M_1 \# M_2 := (M_1 \setminus h_1(0)) \sqcup (M_2 \setminus h_2(0)) / \sim$$

with $h_1(x) \sim h_2(\frac{x}{\|x\|^2})$.

It has been proved by Cerf (see [6]) that $M_1 \# M_2$ is well defined up to orientation-preserving diffeomorphism, and the connected sum of two topological spheres is a topological sphere. If dim $M = 4n - 1$, one has the isomorphism $H^{2n}(B_1 \# B_2, M_1 \# M_2) \to H^{2n}(B_1, M_1) \oplus H^{2n}(B_2, M_2)$ which gives you

$$\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$$

Since, for example, we know $\lambda(M_3) \equiv 1$, then

$$S^7, M_3, M_3 \# M_3, \ldots, \underbrace{M_3 \# \ldots \# M_3}_{6 \text{ times}}$$

all belong to distinct diffeomorphism classes.

This invariant λ can be generalized quite easily to higher dimensions, (see [5]) and allows you to distinguish between a large number of differentiable structures on spheres (e.g. one can find 1414477 structures on the 23-sphere), but it turns out to be too strong to distinguish between all possible structures. In 1963, J.Milnor and M.Kervaire completed the classification in dimension higher than 4, proving that there is only a *finite number* of differentiable structures on every n-dimensional sphere and reducing the computation of this number to some classical problems in algebraic topology.

Their solution relies heavily on the Generalized Poincare' conjecture which had been meanwhile proved in dimension ≥ 5 by S.Smale. What Milnor and Kervaire really classify are h-cobordism classes of homotopy spheres:

Definition 8.2. *Two oriented manifolds M_1 and M_2 are h-cobordant if their disjoint union is the boundary of a manifold W which induces the given orientation on M_1 and the opposite one on M_2 and such that both M_1 and M_2 are deformation retracts of W.*

Now, the h-cobordism classes of manifolds of dimension n form an abelian group under the connected sum operation. The neutral element is the standard S^n, while the inverse of a manifold is the same manifold with opposite orientation. Such a group is denoted by Θ_n.

For $n \geq 5$, S.Smale also proved that two homotopy n-spheres are h-cobordant iff they are diffeomorphic. Hence the classification problem is reduced to computing Θ_n. The method used by Milnor and Kervaire consists of splitting Θ_n in two parts:

Definition 8.3. *A manifold M is* parallelizable *if its tangent bundle is trivial, and it is* almost parallelizable *if there exists a finite set F such that $M \setminus F$ is parallelizable. The set of all h-cobordism classes of n-manifolds which are boundaries of parallelizable manifolds is denoted by bP_{n+1}.*

One has to check that bP_{n+1} is actually a group, and then study separately bP_{n+1} and the quotient Θ_n/bP_{n+1}.

1. For Θ_n/bP_{n+1}, it turns out that there is an inclusion

$$\frac{\Theta_n}{bP_{n+1}} \to \frac{\Pi_n}{J_n(\pi_n(SO))}$$

   where $\Pi_n = \lim_{k\to\infty} \pi_{n+k}(S^k)$ is the n^{th} stable homotopy group of spheres and $J_n : \pi_n(SO) \to \Pi_n$ is the Hopf-Whitehead homomorphism (see [2], chap.3). Since Serre proved Π_n is a finite group, Θ_n/bP_{n+1} is also finite: for $n = 7$, J_7 is known to be surjective, hence $\Theta_7/bP_8 = \{0\}$.

2. The group bP_{n+1} is always finite cyclic: it is trivial for n even and it has order 1 or 2 for $n \equiv 2 \pmod 4$. For $n = 4m$, it is determined by the signature: let σ_m be the smallest positive value of $\sigma(W)$ as W varies among almost-parallelizable manifolds of dim $4m$ without boundary. If M is a homotopy $4m-1$-sphere which is the boundary of the parallelizable manifold B, one can send

$$M \mapsto \sigma(B) \pmod{\sigma_m}$$

   and get a homomorphism $bP_{4m} \to \mathbb{Z}/\sigma_m\mathbb{Z}$ which turns out to be injective, hence bP_{4m} is cyclic. The image has actually order $\sigma_m/8$, and Milnor and Kervaire compute

$$\sigma_m = 2^{2m-1}(2^{2m-1}-1)\frac{B_m a_m j_m}{m}$$

   where B_m is the m^{th} Bernoulli number, $a_m = 1$ for m even and 2 for m odd, and j_m is the order of the image of the Hopf-Whitehead homomorphism J_{4m-1}. Adams proved that j_{4m-1} is the denominator of $B_m/4m$. For $m = 2$, $B_2 = 1/30$, so $j_2 = 240$ and therefore

$$|bP_8| = |\sigma_2/8| = 28$$

   giving exactly 28 differentiable structures on the 7-sphere.

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