GENERALIZATIONS OF DOUADY'S MAGIC FORMULA

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Abstract. We generalize a combinatorial formula of Douady from the main cardioid to other hyperbolic components H of the Mandelbrot set, constructing an explicit piecewise linear map which sends the set of angles of external rays landing on H to the set of angles of external rays landing on the real axis.

In the study of the dynamics of the quadratic family $f_c(z) := z^2 + c$, $c \in \mathbb{C}$, a central object of interest is the *Mandelbrot set*

$$\mathcal{M} := \{c \in \mathbb{C} \ : \ (f_c^{(n)}(0))_{n \geq 0} \text{ is bounded}\}.$$

The Mandelbrot set contains two notable smooth curves: namely, the intersection with the real axis $\mathcal{M} \cap \mathbb{R} = [-2, 1/4]$, and the *main cardioid* of $\mathcal{M}$, which is the set of parameters c for which f_c has an attracting or indifferent fixed point in c (of course, this is smooth except at the cusp $c = 1/4$).

Let $\mathcal{R}$ denote the set of angles of external rays landing on the real axis, and Ξ the set of angles of external rays landing on the main cardioid. The following "magic formula" is due to Douady (for a proof see [Bl]):

Theorem 1 (Douady)**.** *The map*

$$T(\theta) := \begin{cases} \frac{1}{2} + \frac{\theta}{4} & \text{if } 0 \leq \theta < \frac{1}{2} \\ \frac{1}{4} + \frac{\theta}{4} & \text{if } \frac{1}{2} < \theta \leq 1 \end{cases}$$

sends Ξ into $\mathcal{R}$.

In this note, we prove the following generalization of Theorem 1 to other hyperbolic components. Let $D(\theta) := 2\theta \mod 1$ denote the doubling map.

Theorem 2. *Let H be a hyperbolic component which belongs to a vein V of the Mandelbrot set, not in the $1/2$-limb. Let $\mathbf{A}_H$, $\mathbf{B}_H$ be the binary expansions of the root of H, and let δ_V be the complexity of V. Then the map*

$$\Phi_H(\theta) := D^{\delta_V}(\mathbf{B}_H \mathbf{A}_H \cdot \theta)$$

sends the set of external angles of rays landing on the upper part of H into the set $\mathcal{R}$ of angles of rays landing on the real axis.

To illustrate such a formula, let us consider the "kokopelli" component of period 4, which has root of angles $\theta_1 = 3/15 = .\overline{0011}$, $\theta_2 = 4/15 = .\overline{0100}$, thus $\mathbf{A}_H = 0011$,

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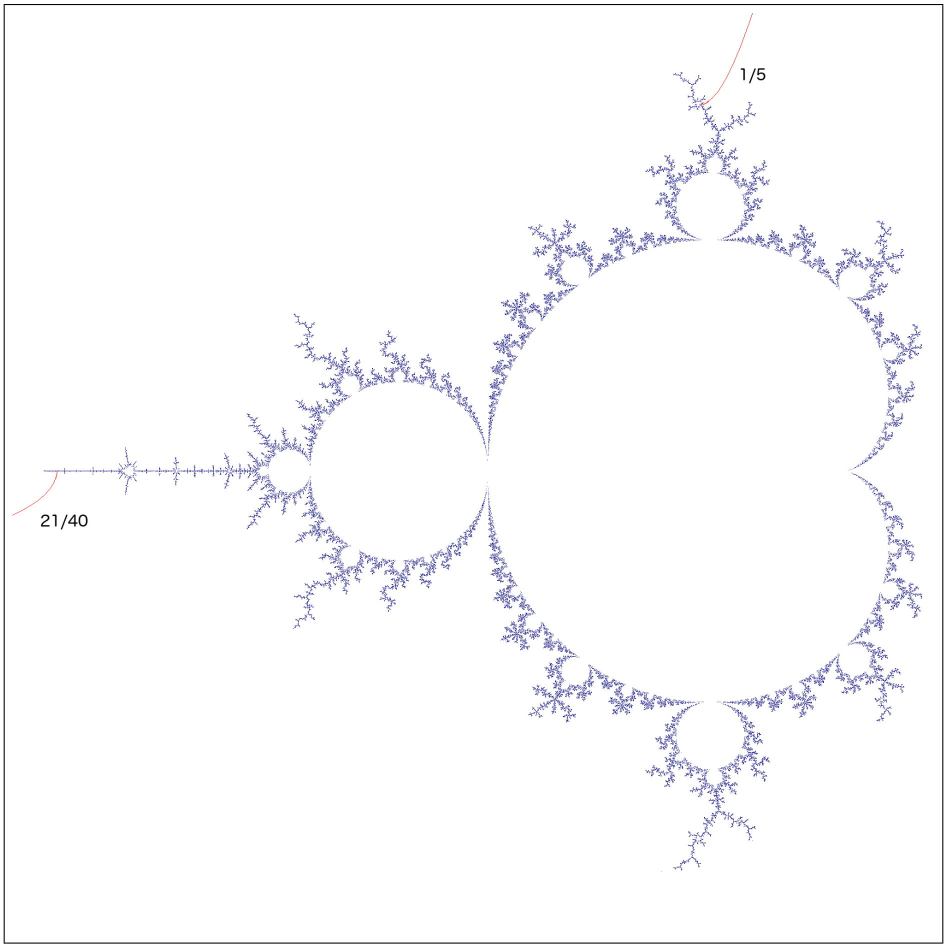

FIGURE 1. The magic formula for the kokopelli component. The angle $\theta = 1/5$ which lands at the root of its component is sent to $\Phi_H(\theta) = 21/40$ which lands on the real axis.

$\mathbf{B}_H = 0100$. This component lies on the principal vein with tip $\theta = 1/4$, hence its complexity is $\delta_V = 1$. Thus,

$$\Phi_H(\theta) := 1000011 \cdot \theta = \frac{67}{128} + \frac{\theta}{128}.$$

Let us note that if H is the main cardioid, then $\delta_V = 0$, $\mathbf{A}_H = 0$, $\mathbf{B}_H = 1$, hence we recover Douady's original formula from Theorem 1.

Remarks. Note that the map Φ_H is very far from being surjective: indeed, the set of angles of rays landing on a hyperbolic component has Hausdorff dimension zero, while the set $\mathcal{R}$ has dimension 1.

During the preparation of this paper we have been informed of the generalization of Douady's formula by Blé and Cabrera [BC]. Let us remark that our formula is quite different, as the one in [BC] does not map into the real axis but rather into the

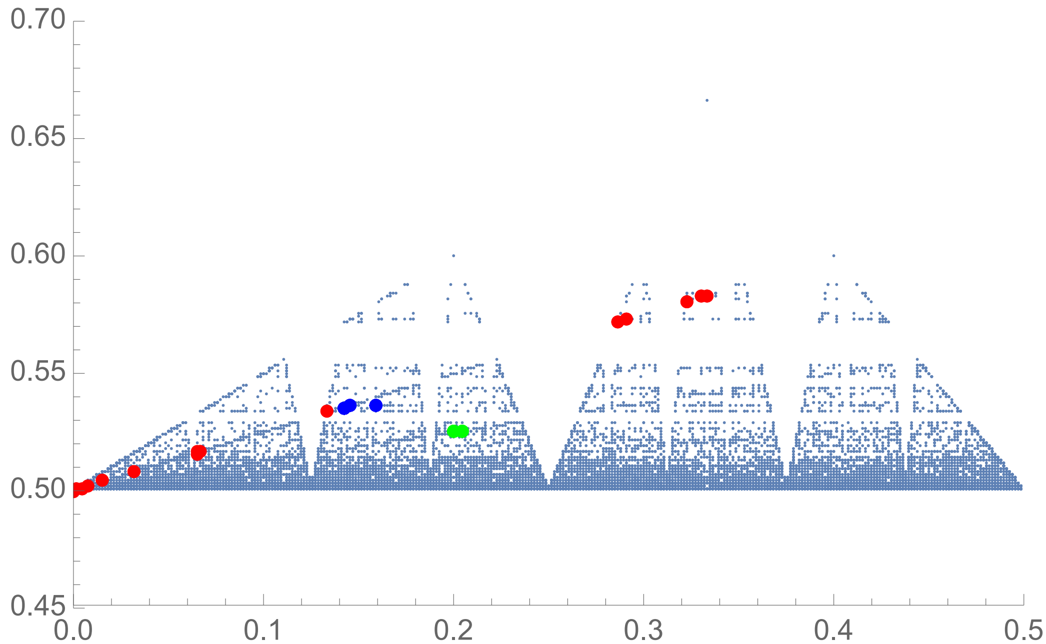

FIGURE 2. The graph of the function $\Psi(x)$ discussed in Remark 4. The red dots correspond to points on the graph of Douady's magic formula function $T = \varphi_H$ for the main cardioid, the (thick) blue dots to the image of φ_H for the rabbit component, and the green dots to the kokopelli component.

tuned copies of the real axis inside small Mandelbrot sets. In fact, we will describe their formula and how it relates to ours in Section 1.2.

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1. Combinatorics of external rays

1.1. **Real rays and the original formula.** Consider the Riemann map $\Phi_M : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus \mathcal{M}$, which is unique if we normalize it so that $\Phi_M(\infty) = \infty$, $\Phi'_M(\infty) = 1$. For each $\theta \in \mathbb{R}/\mathbb{Z}$, we define the *external ray* of angle θ as

$$R_M(\theta) := \{\Phi_M(re^{2\pi i\theta}) \ : \ r > 1\}.$$

The ray $R_M(\theta)$ *lands* if there exists $\gamma(\theta) := \lim_{r\to 1^+} \Phi_M(re^{2\pi i\theta})$. It is conjectured that the ray $R_M(\theta)$ lands for any $\theta \in \mathbb{R}/\mathbb{Z}$; to circumvent this issue, we define as $\widehat{R}_M(\theta)$ the impression of $R_M(\theta)$, and the set of *real angles* as

$$\widehat{\mathcal{R}} := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ \widehat{R}_M(\theta) \cap \mathbb{R} \neq \emptyset\}.$$

Conjecturally, this is exactly the set of angles for which the corresponding ray lands on the real axis. For further details, see e.g. [Za].

The following lemma gives a characterization of $\widehat{\mathcal{R}}$ in terms of the dynamics of the doubling map.

Lemma 3 ([Do])**.** *The set of real angles equals*

$$\widehat{\mathcal{R}} := \{\theta \in \mathbb{R}/\mathbb{Z} \ : \ |D^n(\theta) - 1/2| \geq |\theta - 1/2| \quad \forall n \in \mathbb{N}\}.$$

Proof of Theorem 1. By symmetry, we can assume $0 < \theta < \frac{1}{2}$. Since the external ray of angle θ lands on the main cardioid, then the forward orbit $P := (D^n(\theta))_{n\in\mathbb{N}}$ of the angle θ does not intersect the half-circle $I = (\frac{\theta}{2} + \frac{1}{2}, \frac{\theta}{2})$ (the one which contains 0) (see [BS]). The preimage of I by the doubling map is the union of two intervals $(-\frac{\theta}{4}, \frac{\theta}{4}) \cup (\frac{1}{2} - \frac{\theta}{4}, \frac{1}{2} + \frac{\theta}{4})$. In particular, the forward orbit of θ does not intersect $J = (\frac{1}{2} - \frac{\theta}{4}, \frac{1}{2} + \frac{\theta}{4})$. Thus, if we look at the orbit of $\theta' = T(\theta)$, we have $2\theta' = \frac{\theta}{2} \notin J$ and $D^n(\theta') \in P$ for any $n \geq 2$, hence the forward orbit of θ' does not intersect J, hence θ' is a real angle by Lemma 3. □

Remark 4. By Lemma 3, one recognizes that a way to map any angle $\theta \in \mathbb{R}/\mathbb{Z}$ to the set of real angles is to consider the function

$$\Psi(x) := \frac{1}{2} + \inf_{k\geq 0} \left| D^k(x) - \frac{1}{2} \right|,$$

which indeed satisfies $\Psi(\mathbb{R}/\mathbb{Z}) \subseteq \widehat{\mathcal{R}}$. As you can see from Figure 2, this function is discontinuous, while its graph "contains" the graphs of all the magic formula functions φ_H given by Theorem 2 (which are indeed continuous) for all components H.

1.2. **Tuning and the Blé-Cabrera magic formula.** Given any hyperbolic component H in the Mandelbrot set, let us recall that there is a *tuning map* which sends the main cardioid to the hyperbolic component H, and the Mandelbrot set to a small copy of itself which contains H.

In order to define the map precisely, let $\mathbf{a}_H = .\overline{a_1 \dots a_k} < \mathbf{b}_H = .\overline{b_1 \dots b_k}$ denote the two external angles of rays landing at the root of H, and denote $\mathbf{A}_H = a_1 \dots a_k$ and $\mathbf{B}_H = b_1 \dots b_k$ the two corresponding finite binary words. Moreover, denote $\mathbf{a}'_H = .\overline{a_1 \dots a_k b_1 \dots b_k}$ and $\mathbf{b}'_H = .\overline{b_1 \dots b_k a_1 \dots a_k}$.

The tuning map $\mathfrak{T}_H$ on the set of external angles is now defined as follows. If $\theta = .\epsilon_1\epsilon_2 \dots$ is the binary expansion of θ, then the angle $\mathfrak{T}_H(\theta)$ has binary expansion

$$\mathfrak{T}_H(\theta) = .A_{\epsilon_1} A_{\epsilon_2} \dots$$

where $A_0 = \mathbf{A}_H$, $A_1 = \mathbf{B}_H$. Then, the set $\Xi_H := \mathfrak{T}_H(\Xi)$ is the set of rays landing on the boundary of H.

If $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ has infinite binary expansion (i.e. it is not a dyadic number) and $S = s_1 \dots s_n$ is a finite word on the alphabet $\{0, 1\}$, we denote by $S \cdot \theta$ the element of $\mathbb{T}$

$$S \cdot \theta := \sum_{k=1}^{n} s_k 2^{-k} + 2^{-n}\theta$$

i.e. the point whose binary expansion is the concatenation of S and the binary expansion of θ. Recall that tuning behaves well with respect to concatenation, namely for any S, H and θ we have

$$\mathfrak{T}_H(S \cdot \theta) = \mathfrak{T}_H(S) \cdot \mathfrak{T}_H(\theta).$$

We define the *small real vein* associated to H, and denote it as $\mathbb{R}_H$, to be the real vein of the small Mandelbrot set associated to H. Let us denote as $\mathcal{R}_H$ the set of external angles of rays landing on $\mathbb{R}_H$, and as $\widehat{\mathcal{R}}_H$ the set of external angles of rays whose impression intersects $\mathbb{R}_H$. Clearly, $\mathbb{R}_{H_0} = \mathbb{R}$ if H_0 is the main cardioid, and $\mathcal{R}_{H_0} = \mathcal{R}$.

Proposition 5 (Blé-Cabrera [BC]). *The map $T_H := \mathfrak{T}_H \circ T \circ \mathfrak{T}_H^{-1}$ maps Ξ_H into the small real vein $\widehat{\mathcal{R}}_H$. Moreover, this map (restricted to Ξ_H) is piecewise affine: in fact, it can be written in terms of binary expansions as*

$$T_H(\theta) = \mathbf{B}_H \mathbf{A}_H \cdot \theta$$

if $\theta \in (\mathbf{a}_H, \mathbf{a}'_H)$, and

$$T_H(\theta) = \mathbf{A}_H \mathbf{B}_H \cdot \theta$$

if $\theta \in (\mathbf{b}'_H, \mathbf{b}_H)$.

Proof. By construction, the tuning map $\mathfrak{T}_H : \Xi \setminus \{0\} \to \Xi_H \setminus \{\mathbf{a}_H, \mathbf{b}_H\}$ is a bijection, so the first statement follows by looking at the diagram:

$$\begin{array}{ccc} \Xi & \xrightarrow{T} & \widehat{\mathcal{R}} \\ \downarrow{\scriptstyle \mathfrak{T}_H} & & \downarrow{\scriptstyle \mathfrak{T}_H} \\ \Xi_H & \xrightarrow{T_H} & \widehat{\mathcal{R}}_H. \end{array}$$

Moreover, since $T(\theta) = 10 \cdot \theta$ for $\theta < 1/2$ and tuning behaves nicely with respect to concatenation, we have

$$T_H(\theta) = \mathfrak{T}_H(10 \cdot \mathfrak{T}_H^{-1}(\theta)) = \mathfrak{T}_H(10) \cdot \theta$$

which proves the second claim. □

1.3. **Veins, pseudocenters and complexity.** Given a dyadic number $\theta \in S^1 = \mathbb{R}/\mathbb{Z}$, we define its *complexity* as

$$\|\theta\| := \min\{k \geq 0 \ : \ D^k(\theta) = 0 \mod 1\}.$$

Of course, if $\theta = \frac{p}{2^q}$ with p odd, then $\|\theta\| = q$. Given an interval (θ^-, θ^+) with $\theta^- < \theta^+$, we define its *pseudocenter* θ_0 to be the dyadic rational of lowest complexity inside the interval (θ^-, θ^+).

A pair of elements (θ^-, θ^+) in $\mathbb{T}$ is a *ray pair* if the two external rays of angle θ^- and θ^+ combinatorially land at the same parameter on the boundary of the Mandelbrot set. To be precise, one starts by defining a relation on $\mathbb{Q}/\mathbb{Z}$ by setting that $\theta_1 \sim_{\mathbb{Q}} \theta_2$ if $R_M(\theta_1)$ and $R_M(\theta_2)$ land at the same point. Then, one takes the transitive closure of this relation and finally its topological closure to define an equivalent relation $\sim$ on $\mathbb{R}/\mathbb{Z}$. As constructed by Thurston [Th], there is a lamination QML on the unit disk such that its induced equivalent relation is precisely $\sim$. If MLC holds, then $\theta_1 \sim \theta_2$ if and only it $R_M(\theta_1)$ and $R_M(\theta_2)$ land on the same point.

Ray pairs are partially ordered: in fact, we say $(\theta_1^-, \theta_1^+) \prec (\theta_2^-, \theta_2^+)$ if the leaf (θ_1^-, θ_1^+) separates (θ_2^-, θ_2^+) from 0. Let us denote as $N(\theta)$ the number of ends of the Hubbard tree associated to the angle θ. Recall that if $\theta_1 \prec \theta_2$, then $N(\theta_1) \leq N(\theta_2)$.

A dyadic number θ_0 defines a combinatorial vein in the Mandelbrot set as follows.

Definition 6. *Given a dyadic rational number θ_0, we define the* combinatorial vein *of θ_0 as the set of ray pairs (θ_1, θ_2) such that:*

(1) *θ_0 is the pseudocenter of (θ_1, θ_2);*
(2) *$N(\theta_0) = N(\theta_1)$.*

For instance, if $\theta_0 = \frac{1}{4}$, then the vein extends all the way to $(\frac{1}{7}, \frac{2}{7})$, since $N(\frac{1}{4}) = 3$, and one can check easily that $N(\frac{1}{7}) = 3$ (the "rabbit"). On the other hand, if $\theta_0 = \frac{3}{16}$, then the vein extends up to $(\frac{39}{224}, \frac{43}{224})$. Indeed, $N(\frac{3}{16}) = 5$, and $N(\theta) = 3$ if $\theta \in (\frac{1}{7}, \frac{39}{224})$.

Definition 7. *The* complexity *of a vein V with pseudocenter θ_0 is given by*

$$\delta_V = \|\theta_0\| - 1.$$

The complexity δ_V equals:

- the smallest k such that $D^k(\theta_0) = 1/2$;
- the smallest k such that $D^k(\theta^-)$ and $D^k(\theta^+)$ lie on opposite connected components of the set $\mathbb{T} \setminus \{0, 1/2\}$;
- the smallest k such that $f_c^k(c)$ lies on the spine $[-\beta, \beta]$ of the Julia set of f_c for c which belongs to the vein.

Clearly, $\delta_V = 0$ if and only if V is the real vein. Given a vein V in the upper half plane (i.e. with $\theta_0 < 1/2$), its *lower side* is the set of angles θ^+ for which there exists $\theta^- < \theta^+$ such that (θ^-, θ^+) belongs to the vein.

Proposition 8. *Let V be a vein in the upper half plane, and δ_V be its complexity. Then for each ray pair (θ^-, θ^+) which belongs to V we have*

$$D^{\delta_V}(\theta^+) \subseteq \widehat{\mathcal{R}}.$$

In order to prove the proposition, we need the following

Lemma 9. *Let (θ^-, θ^+) be a ray pair, and θ_0 be its pseudocenter, with $\theta_0 < 1/3$ (i.e. the hyperbolic component lies in the upper half plane, and not in the $1/2$ limb). Then we have*

$$\theta^+ - \theta_0 < \theta_0 - \theta^-.$$

Proof. Recall that, for each rational $\frac{p}{q}$ with p, q coprime, there exists a unique set $C_{p/q}$ of q points on $\mathbb{T}$ such that the doubling map acts on $C_{p/q}$ with rotation number p/q, i.e. $D(x_i) = x_{i+p}$, where the indices are considered modulo q. We know that the set $C_{p/q}$ is precisely the set of external rays which land on the α fixed point for f_c, when c belongs to the p/q-limb. Let us call *sectors* the q connected components of $\mathbb{T} \setminus C_{p/q}$. In particular, we call *critical sector* $\mathcal{S}_1$ the smallest sector, and *central sector*

$\mathcal{S}_0$ the largest sector. We say an interval $I \subseteq \mathbb{T}$ is *embedded* if it is entirely contained within one sector. Let us consider the interval $I_0 = [\theta^-, \theta^+]$, which is embedded by construction.

Let $k \geq 0$ be the smallest integer such that $I_{k+1} := D^{k+1}(I_0)$ is not embedded (it must exist, since D doubles lengths, and the length of an embedded interval is bounded above by a constant $c < 1$). Note that the doubling map is a homeomorphism between each non-central sector and its image, so I_k must be contained in the central sector.

Let us now consider the set $\widehat{C}_{p/q} = C_{p/q} + 1/2$, which is a subset of the central sector and equals the set of external rays which land on $-\alpha$, the preimage of the α fixed point. Thus, let us call *subsectors* the connected components of $\mathcal{S}_0 \setminus \widehat{C}_{p/q}$. In particular, exactly two subsectors $\mathcal{S}^-$ and $\mathcal{S}^+$ map to the critical sector $\mathcal{S}_1$; let us call them *central subsectors*. Since the rays $D^k(\theta^-)$ and $D^k(\theta^+)$ land together, then either they are both contained in the same subsector, or one is contained in one central subsector and the other one in the other. If they are contained in the same subsector, then the whole I_k is contained in the same subsector, hence its image I_{k+1} is still embedded, contradicting the definition of k. Thus, the two endpoints of I_k are contained in two distinct central subsectors. Since I_k is embedded, then it must contain 0; moreover, since 0 is the dyadic of lowest possible complexity, this means that $D^k(\theta_0) = 0$. Now, by construction the images $D^{k+1}(\theta^-)$ and $D^{k+1}(\theta^+)$ lie in the critical sector $\mathcal{S}_1$, which is contained inside the arc $[0, 1/2]$ since the hyperbolic component lies in the upper half plane. This means that

$$\ell([0, D^{k+1}(\theta^+)]) < 1/2 < \ell([D^{k+1}(\theta^-), 0])$$

(where ℓ denotes the length of the intervals) and, since D^{k+1} is a homeomorphism when restricted to $[\theta_0, \theta^+]$ and to $[\theta^-, \theta_0]$,

$$2^{k+1}\ell([\theta_0, \theta^+]) = \ell(D^{k+1}[\theta_0, \theta^+]) = \ell([0, D^{k+1}(\theta^+)]) < 1/2$$

and similarly

$$2^{k+1}\ell([\theta^-, \theta_0]) = \ell(D^{k+1}[\theta^-, \theta_0]) = \ell([D^{k+1}(\theta^-), 0]) > 1/2$$

hence by comparing the previous two equations

$$\ell([\theta_0, \theta^+]) < \ell([\theta^-, \theta_0])$$

which proves the claim. □

1.4. **Combinatorial Hubbard trees.** Recall that every angle $\theta \in \mathbb{T}$ has an associated lamination on the disc which is invariant by the doubling map (see [Th]). The (two) longest leaves of the lamination are called *major leaves*, and their common image is called *minor leaf* and will be denoted by m. Moreover, we let β denote the leaf $\{0\}$, which we will take as the root of the lamination (the notation is due to the fact that the ray at angle 0 lands at the β-fixed point). The dynamics on the lamination is induced by the dynamics of the doubling map on the boundary circle. In particular, let us denote $f : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ to be a continuous function on the filled-in disc which extends the doubling map on the boundary S^1. Let us denote by Δ the diameter of the circle which connects the boundary points at angles $\theta/2$ and $(\theta+1)/2$. Then we shall also choose f so that it maps homeomorphically each connected component of $\mathbb{D} \setminus \Delta$ onto $\mathbb{D}$.

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two distinct leaves. Then we define the *combinatorial segment* $[\mathcal{L}_1, \mathcal{L}_2]$ as the set of leaves $\mathcal{L}$ of the lamination which separate $\mathcal{L}_1$ and $\mathcal{L}_2$. Some simple properties of combinatorial segments are the following:

(a) if $\mathcal{L} \in [\mathcal{L}_1, \mathcal{L}_2]$, then $[\mathcal{L}, \mathcal{L}_1] \subseteq [\mathcal{L}_1, \mathcal{L}_2]$;
(b) for any choice of leaves $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3$ we have $[\mathcal{L}_1, \mathcal{L}_2] \subseteq [\mathcal{L}_3, \mathcal{L}_1] \cup [\mathcal{L}_3, \mathcal{L}_2]$;
(c) the image of $[\mathcal{L}_1, \mathcal{L}_2]$ equals:

$$f([\mathcal{L}_1, \mathcal{L}_2]) = \begin{cases} [f(\mathcal{L}_1), f(\mathcal{L}_2)] & \text{if } \Delta \text{ does not separate } \mathcal{L}_1 \text{ and } \mathcal{L}_2, \\ [f(\mathcal{L}_1), m] \cup [f(\mathcal{L}_2), m] & \text{if } \Delta \text{ separates } \mathcal{L}_1 \text{ and } \mathcal{L}_2. \end{cases}$$

(d) in any case, for any leaves $\mathcal{L}_1, \mathcal{L}_1$ we have

$$[f(\mathcal{L}_1), f(\mathcal{L}_2)] \subseteq f([\mathcal{L}_1, \mathcal{L}_2]) \subseteq [f(\mathcal{L}_1), m] \cup [f(\mathcal{L}_2), m].$$

Finally, we shall say that a set S of leaves is *combinatorially convex* if whenever $\mathcal{L}_1$ and $\mathcal{L}_2$ belong to S, then the whole set $[\mathcal{L}_1, \mathcal{L}_2]$ is contained in S.

We now define

$$H_n := \bigcup_{0 \leq i \leq n} [\beta, f^i(m)]$$

and

$$H := \bigcup_{n \in \mathbb{N}} H_n.$$

We call the set H the *combinatorial Hubbard tree* of f, as it is a combinatorial version of the (extended) Hubbard tree.

Lemma 10. *The combinatorial Hubbard tree H has the following properties.*

(1) *The set H is the smallest combinatorially convex set of leaves which contains β, m and is forward invariant.*
(2) *Let $N \geq 0$ be an integer such that $f^{N+1}(m) \in H_N$. Then we have $H = H_N$.*

Note that $N+2$ coincides with the number of ends of the extended Hubbard tree.

Proof. (1) Let us check that H is combinatorially convex. Suppose $\mathcal{L}_1, \mathcal{L}_2$ belong to H, so that say $\mathcal{L}_1 \in [\beta, f^i(m)]$ and $\mathcal{L}_2 \in [\beta, f^j(m)]$. Then

$$[\mathcal{L}_1, \mathcal{L}_2] \subseteq [\mathcal{L}_1, \beta] \cup [\beta, \mathcal{L}_2] \subseteq [f^i(m), \beta] \cup [\beta, f^j(m)] \subseteq H$$

as required. In order to check that H is forward invariant, note that by point (d) and using that $f(\beta) = \beta$ yields the inclusion

$$f([\beta, f^i(m)]) \subseteq [f(\beta), m] \cup [m, f^{i+1}(m)] \subseteq [\beta, m] \cup [\beta, f^{i+1}(m)] \tag{1}$$

and the right-hand side is contained in H by construction, so $f(H) \subseteq H$.

In order to check the minimality, let S be a combinatorially convex, forward invariant set of leaves which contains β and m. Then by convexity S contains $[\beta, m]$. Moreover, by forward invariance for any $i \geq 0$ we have

$$S \supseteq f^i([\beta, m]) \supseteq [f^i(\beta), f^i(m)] = [\beta, f^i(m)]$$

thus $S \supseteq H$.

(2) Note that by construction $H_N \subseteq H$, and the same proof as in (1) shows that H_N is combinatorially convex. In order to prove the claim it is thus enough to check that H_N is forward invariant, because then by minimality we get $H_N \supseteq H$ and the claim is proven. To prove forward invariance, note that

$$f(H_N) = \bigcup_{0 \le i \le N} f([\beta, f^i(m)]) \subseteq H_N \cup [\beta, f^{N+1}(m)]$$

and since $f^{N+1}(m) \in H_N$ then $[\beta, f^{N+1}(m)] \subseteq H_N$, proving the claim. □

Lemma 11. *Let $m_1 < m_2$, and H_1, H_2 be the corresponding combinatorial Hubbard trees. Then*

$$H_1 \subseteq H_2.$$

As a corollary,

$$\#Ends(T_1) \le \#Ends(T_2).$$

Proof. By definition, $m_1 < m_2$ means that $m_1 \in [\beta, m_2]$. Thus, $m_1 \in H_2$ and since H_2 is forward invariant we have $f^i(m_1) \in H_2$ for any $i \ge 0$. Since also $\beta \in H_2$ and H_2 is combinatorially convex, then $[\beta, f^i(m_1)] \subseteq H_2$ for any $i \ge 0$, thus $H_1 \subseteq H_2$ as required. For the corollary, note that since the trees are dual to the laminations, $T_1 \subseteq T_2$, and in general a connected subtree of a tree cannot have more ends than the ambient tree. □

Lemma 12. *Let (θ^-, θ^+) be a ray pair, and θ_0 its pseudocenter, with $\|\theta_0\| = q$. Then*

$$\#Ends(T_{\theta^+}) = \#Ends(T_{\theta_0})$$

if and only if the arcs $I_k = (D^k(\theta^-), D^k(\theta^+))$ for $k = 0, \dots, q-1$ are disjoint.

Proof. Note that the forward orbit of θ_0 has cardinality $q+1$, and since its minor leaf is a point (and so are all its forward iterates), the number of ends of T_{θ_0} is also $q+1$. Now, consider the minor leaf $m = (\theta^-, \theta^+)$ and its forward iterates. Note that by Lemma 13 the arc I_k for $k \le q-1$ never contains 0. Thus, the intervals I_j and I_k are disjoint if and only if neither $f^j(m) \in [\beta, f^k(m)]$ nor $f^k(m) \in [\beta, f^j(m)]$. Now, if all the intervals are disjoint, then $f^k(m) \notin [\beta, f^i(m)]$ for any $0 \le i < k \le q-1$, hence the number of ends of T_{θ^+} is at least $q+1$, which implies it is exactly $q+1$ since by the previous Lemma it cannot exceed the number of ends of T_{θ_0}. Vice versa, if $0 \le i < k \le q-1$ and $f^k(m) \notin [\beta, f^i(m)]$, then either I_k is disjoint from I_i, or $f^i(m) \in [\beta, f^k(m)]$. This however is impossible, as it implies $I_k \subseteq I_i$ with $i < k$. Indeed, let us denote as x_k and x_i the pseudocenters of, respectively, I_k and I_i. As the complexity of the pseudocenter decreases precisely by 1 under iteration, since $i < k$ one has $\|x_i\| > \|x_k\|$. However, since $I_k \subseteq I_i$, then $x_k \in I_i$, hence by definition of pseudocenter $\|x_k\| \ge \|x_i\|$, contradicting the previous statement. □

Lemma 13. *Let $[\alpha, \beta]$ be an embedded arc in S^1 which does not contain 0, and θ_0 its pseudocenter, with $q = \|\theta_0\|$. Then for all $0 \le k \le q-1$, the arc $[D^k(\alpha), D^k(\beta)]$ is embedded and does not contain 0.*

Proof. We claim that there exists a minimal k such that $[D^k(\alpha), D^k(\beta)]$ contains $1/2$, and moreover all arcs $[D^h(\alpha), D^h(\beta)]$ for $0 \leq h \leq k$ are embedded. Indeed, the map D doubles lengths of arcs of length less than $1/2$, hence eventually the length of one of its images is more than $1/2$. Let k be minimal such that the length of $J = [D^k(\alpha), D^k(\beta)]$ is at least $1/2$. Note that all previous iterates are embedded arcs, moreover J must contain either 0 or $1/2$. However, by minimality of k, since the preimages of 0 are 0 and $1/2$ and the initial arc does not contain 0, J must contain $1/2$. Since J contains $1/2$ and does not contain 0, then its pseudocenter is $1/2$, and since $J = D^k([\alpha, \beta])$ one has $D^k(\theta_0) = 1/2$, hence $k = q - 1$, which completes the proof of the claim. □

Proof of Proposition 8. Let $k = \delta_V$, and let $\theta^- < \theta^+$ be the two endpoints of the leaf. We need to show that $D^k(\theta^+)$ belongs to $\widehat{\mathcal{R}}$. Let us set $I_0 := (\theta^-, \theta^+)$ and $I_h := D^h(I_0)$. Since the component belongs to V, we have by Lemma 12 that the intervals I_h for $0 \leq h \leq k+1$ are pairwise disjoint. Moreover, by definition of δ_V the interval I_k contains $1/2$. Now if one looks at the lamination it follows that $f^{k+1}(m) \in H_k$, so we are in the hypothesis of Lemma 10. Thus, all higher iterates $f^i(m)$ for $i \geq k+1$ belong to H_k. This means that the leaf $f^k(m)$ separates the point $\{1/2\}$ from all postcritical leaves $f^i(m)$ with $i \geq 0$. Thus, if we consider the orbit $\{D^i(\theta^+), i \geq 0\}$, we have that no iterate lies in the interior of I_k, and points in complement of I_k are by Lemma 9 at least at distance $|D^k(\theta^+) - 1/2|$ from $1/2$, hence the element in the orbit closest to $1/2$ is $D^k(\theta^+)$, so $D^k(\theta^+)$ belongs to $\widehat{\mathcal{R}}$, as required. □

2. Renormalization and landing

We proved that any element in the image of Φ_H combinatorially lands on the real axis. To complete the proof of the main theorem, we need to show it actually lands. In order to do so, we will prove that it is not renormalizable, hence it lands by Yoccoz's theorem.

Proposition 14. *Let H be a hyperbolic component which does not lie in the $1/2$-limb, and let $\theta \in \mathbb{R}/\mathbb{Z}$ be an irrational angle of an external ray which lands on the boundary of H. Then the external ray at angle $\Phi_H(\theta)$ is not renormalizable, hence the corresponding ray lands.*

The proof uses the concept of maximally diverse sequence from [Sh], which we discuss in the following section.

2.1. Maximally diverse sequences.

Definition 15. *A sequence $(s_n) \in \{0,1\}^{\mathbb{N}}$ is* maximally diverse *if the subsequences*

$$(s_{i+np})_{n \in \mathbb{N}}$$

with $p \geq 1$, $0 \leq i \leq p-1$ are all distinct.

Lemma 16. *A sequence (s_n) is maximally diverse if and only if the subsequences*

$$(s_{i+np})_{n\in\mathbb{N}}$$

with $p \geq 1$, $i \geq 0$ are all distinct.

Proof. Suppose that the two subsequences $(s_{i+np})_{n\in\mathbb{N}}$ and $(s_{j+nq})_{n\in\mathbb{N}}$ with $p, q \geq 1$ and $i, j \geq 0$ are equal. Then for any $k \geq 1$ we also have that $(s_{i+npk})_{n\in\mathbb{N}}$ and $(s_{j+nqk})_{n\in\mathbb{N}}$ are equal. Now, we can choose k large enough so that $i < pk$ and $j < qk$; then s is not maximally diverse by definition. □

Corollary 17. *If s is maximally diverse, then for any $i \geq 0$ and $p \geq 1$ the subsequence*

$$(s_{i+np})_{n\in\mathbb{N}}$$

is maximally diverse.

Lemma 18. *Let $s \in \{0,1\}^{\mathbb{N}}$ and $s' = \sigma(s)$ its shift, i.e. $(s')_n = s_{n+1}$ for any n. Then s is maximally diverse if and only if s' is maximally diverse.*

Proof. If s is not maximally diverse, then there exist $i, j \geq 0$ and $p, q \geq 1$ such that $s_{i+np} = s_{j+nq}$ for all $n \geq 0$. Hence, also $s_{i+p+np} = s_{j+q+nq}$ for all $n \geq 0$. Since $i+p \geq 1$ and $j+q \geq 1$, this also implies $s'_{i+p-1+np} = s'_{j+q-1+nq}$ for all $n \geq 0$, hence s' is not maximally diverse. Vice versa, if s' is not maximally diverse then there exist $i, j \geq 0$ and $p, q \geq 1$ such that $s'_{i+np} = s'_{j+nq}$ for all $n \geq 0$, which implies $s_{i+1+np} = s_{j+1+nq}$ for all $n \geq 0$, hence s is not maximally diverse by the previous Lemma. □

Recall an infinite sequence (ϵ_n) is *Sturmian* if there exists $\alpha \in (0,1) \setminus \mathbb{Q}$, $\beta \in \mathbb{R}$ such that

$$\epsilon_n = \lfloor (n+1)\alpha + \beta \rfloor - \lfloor n\alpha + \beta \rfloor - \lfloor \beta \rfloor$$

for all $n \geq 0$.

Theorem 19 (Shallit [Sh])**.** *Sturmian sequences are maximally diverse.*

2.2. **Proof of Proposition 14.** Let θ be an external angle of a ray landing on the boundary of the hyperbolic component H, of period p. If θ is irrational, then θ is the tuning of an irrational angle η of a ray landing on the main cardioid. Thus, the binary expansion of θ is

$$\theta := S_{\epsilon_0} S_{\epsilon_1} S_{\epsilon_2} \dots$$

where (S_0, S_1) are the binary expansions of the two rays landing at the root of H, and (ϵ_n) is the binary expansion of the angle of a ray landing on the main cardioid. Thus, by [BS] the sequence (ϵ_n) is a Sturmian sequence, hence by Theorem 19 it is a maximally diverse sequence.

Hence, for any $\alpha = 0, \dots, p-1$ we have that either $(\theta_{\alpha+np})_{n\in\mathbb{N}}$ is maximally diverse (if $(S_0)_\alpha \neq (S_1)_\alpha$) or is constant (if $(S_0)_\alpha = (S_1)_\alpha$). Thus, if we consider the image $\sigma := \Phi_H(\theta)$ we have by Lemma 18 that either $(\sigma_{\alpha+np})_{n\in\mathbb{N}}$ is maximally diverse or is eventually constant.

Note that since H does not intersect the real axis, then either $(S_0)_1 = (S_1)_1 = 0$ (if H lies in the upper half plane) or $(S_0)_1 = (S_1)_1 = 1$ (if H lies in the lower half plane). In both cases, the subsequence $(\theta_{1+np})_{n\in\mathbb{N}}$ is eventually constant. Thus, there exists $j \geq 1$ such that the subsequence

$$(\sigma_{j+np})_{n\in\mathbb{N}}$$

is eventually constant.

Let us now suppose by contradiction that the angle σ is renormalizable. This implies that there exist two words Z_0, Z_1 with $|Z_0| = |Z_1| = q$ such that $\sigma \in \{Z_0, Z_1\}^{\mathbb{N}}$. Since by construction the angle σ is real, the only possibility is that $(Z_1)_i = 1-(Z_0)_i$ for any $i = 1, \dots, q$.

Case 1. Let us first assume that p is not a multiple of q.

Then let $l := \mathrm{lcm}(p, q)$, and consider all remainder classes modulo l.

Definition 20. *Let us define two remainder classes α, α' modulo p to be q-equivalent if there exists $m \in \{0, \dots, \frac{l}{q}-1\}$ and $\beta, \beta' \in \{0, \dots, q-1\}$ such that $\alpha \equiv qm + \beta \mod p$ and $\alpha' \equiv qm + \beta' \mod p$.*

Lemma 21. *Let $p, q \geq 1$ be two integers, with p not a multiple of q. Now, suppose that a set $A \subseteq \{0, \dots, p-1\}$ is not empty and has the following property: if $\alpha \in A$ and $\alpha' \in A$ is q-equivalent to α, then $\alpha' \in A$. Then $A = \{0, \dots, p-1\}$.*

Proof. If $p \leq q$, the claim is almost trivial: indeed, the set $\{0, \dots, q-1\}$ projects to all possible classes modulo p. Let us suppose now $p > q$, and let $k := \lfloor \frac{p}{q} \rfloor$. Then each interval $A_m := [qm, qm+q-1]$ with $0 \leq m \leq k$ lies in some equivalence class. Moreover, if we choose $j \in \{0, \dots, p-1\}$ such that $j \equiv (k+1)q \mod p$, then also each interval $B_m := [qm+j, qm+j+q-1]$ with $0 \leq m \leq k$ lies in some equivalence class. Then, note that B_m intersects both A_m and A_{m+1}, hence all remainder classes must belong to the same equivalence class. □

Now, let us pick $\alpha \in \{0, \dots, p-1\}$ such that $(\sigma_{\alpha+np})$ is eventually constant. Find γ in $\{0, \dots, l-1\}$ such that $\gamma \equiv \alpha \mod p$, and let $\beta \in \{0, \dots, q-1\}$ such that $\beta \equiv \gamma \mod q$. Then one can write $\gamma = \beta + mq$ with $0 \leq m \leq \frac{l}{q} - 1$.

Suppose that one has

$$\sigma = Z_{\epsilon_0} Z_{\epsilon_1} \dots$$

with $\epsilon_i \in \{0, 1\}$.

If for some $\alpha \in \{0, \dots, p-1\}$ the sequence $(\sigma_{\alpha+np})_{n\in\mathbb{N}}$ is eventually constant (e.c.), so is its subsequence $(\sigma_{\gamma+nl})_{n\in\mathbb{N}}$, which coincides with $((Z_{\epsilon_{m+nl/q}})_\beta)_{n\in\mathbb{N}}$. Since $(Z_0)_\beta \neq (Z_1)_\beta$, this implies $(\epsilon_{m+nl/q})_{n\in\mathbb{N}}$ is also eventually constant. On the other hand, if $(\sigma_{\alpha+np})_{n\in\mathbb{N}}$ is not eventually constant, then it is maximally diverse, hence by Lemma 17 so is $(\sigma_{\gamma+nl})_{n\in\mathbb{N}}$, hence also $(\epsilon_{m+nl/q})_{n\in\mathbb{N}}$ is not eventually constant.

Let us now consider another $\alpha' \in \{0, \dots, p-1\}$ which is q-equivalent to α. Then by the above discussion

$$(\sigma_{\alpha+np})_{n\in\mathbb{N}} \text{ e.c. } \Leftrightarrow (\epsilon_{m+nl/q})_{n\in\mathbb{N}} \text{ e.c. } \Leftrightarrow (\sigma_{\alpha'+np})_{n\in\mathbb{N}} \text{ e.c.}$$

By Lemma 21, since we know that there exists at least some α for which $(\sigma_{\alpha+np})_{n\in\mathbb{N}}$ is e.c., then each sequence $(\sigma_{\alpha'+np})_{n\in\mathbb{N}}$ is e.c. for any $\alpha' \in \{0, \dots, p-1\}$, hence σ is also eventually periodic, thus the angle θ cannot be irrational.

Case 2. Finally, let us consider the case when p is a multiple of q.

Then recall one can write

$$\sigma = PS_{\epsilon_0}S_{\epsilon_1}\cdots = Z_{\eta_0}Z_{\eta_1}\ldots$$

where P is a finite word of some length $k \geq 0$.

Now, let us suppose that k is not a multiple of q. Note that since H is contained in either the upper or the lower half plane, we have $(S_0)_1 = (S_1)_1$; moreover, we claim that $(S_0)_p \neq (S_1)_p$: in fact, if one considers the external angles $\theta_0 = .\overline{S_0}$ and $\theta_1 = .\overline{S_1}$, the number of periodic external rays of period which divides p and lie in the interval (θ_0, θ_1) is even, since on each landing point exactly two rays land. Hence, in their binary expansion the last digit of S_0 must be opposite to the last digit of S_1.

Now, for each $i \geq 0$ there is some index j such that Z_{η_j} overlaps with both S_{ϵ_i} and $S_{\epsilon_{i+1}}$. As the last part of Z_i must coincide with the first part of $S_{\epsilon_{i+1}}$, and since S_0 and S_1 start with the same symbol, this forces Z_{η_j} to be either Z_0 or Z_1, independently of i. However, since the last symbols of S_0 and S_1 are different, this means that S_{ϵ_i} is also fixed, hence the sequence (ϵ_i) must be eventually constant, which contradicts the irrationality of θ.

Finally, if k is multiple of q, then S_0 and S_1 are finite concatenations of Z_0, Z_1, which means that θ lies already in the small copy of the Mandelbrot set with roots $\{Z_0, Z_1\}$. Since $\{Z_0, Z_1\}$ represent a real pair, such a small copy of the Mandelbrot set lies in the 1/2-limb, which contradicts the fact that H is outside such limb and completes the proof of Proposition 14.

2.3. **Proof of the magic formula.**

Proof of Theorem 2. If θ belongs to Ξ_H, then by Proposition 5 the angle $\mathbf{B}_H\mathbf{A}_H \cdot \theta$ belongs to the upper part of the combinatorial vein V on which H lies. Hence, by Proposition 8 the angle $\Phi_H(\theta) := D^{\delta_V}(\mathbf{B}_H\mathbf{A}_H \cdot \theta)$ belongs to $\widehat{\mathcal{R}}$. Finally, by Proposition 14 the ray actually lands, hence $\Phi_H(\theta)$ belongs to $\mathcal{R}$, as claimed. □

3. An alternate formula

We conclude with another possible generalization of Douady's formula. This version does not depend on the vein structure of $\mathcal{M}$.

Proposition 22. *Let H be any hyperbolic component, and Θ_H be the set of external angles of rays landing on the small Mandelbrot set with root H (in particular, this set contains the set of external angles landing on the boundary of H). Then there exists an affine map φ_H such that*

$$\varphi_H(\Theta_H) \subseteq \mathcal{R}.$$

Lemma 23. *For each hyperbolic component H of period $p > 1$, one has*

$$\left| D^n(\theta) - \frac{1}{2} \right| \geq \frac{1}{2^{2p}}$$

for all $\theta \in \Theta_H$, for all $n \geq 0$.

Proof. Let $\Sigma_0 = \mathbf{A}_H$ and $\Sigma_1 = \mathbf{B}_H$ be the binary words which give the binary expansion of the two angles landing at the root of H, and let p be the period of H, which equals the length of both Σ_0 and Σ_1. By the construction of tuning operators, any external angle θ landing on the small Mandelbrot copy associated to H has binary expansion of type

$$\theta = .\Sigma_{\epsilon_1}\Sigma_{\epsilon_2}\dots\Sigma_{\epsilon_n}\dots$$

where $\epsilon_i \in \{0,1\}$ for all i. Since H is not the main cardioid, then both Σ_0 and Σ_1 contain both the symbol 0 and the symbol 1. As a consequence, any block of consecutive equal digits in the binary expansion of θ cannot have length larger than $2p-2$. However, all numbers in the interval $U_p = \left(\frac{1}{2} - \frac{1}{2^{2p}}, \frac{1}{2} + \frac{1}{2^{2p}}\right)$ have binary expansion of type either

$$.0\underbrace{1\dots1}_{2p-1} \qquad \text{or} \qquad .1\underbrace{0\dots0}_{2p-1},$$

hence none of the iterates $D^n(\theta)$ can lie in the interval U_p. □

Proof of Proposition 22. The map is given by

$$\varphi_H(\theta) := 0\underbrace{1\dots1}_{2p-1}\cdot\theta$$

In fact, by the above observation, $\varphi_H(\theta) \in U_p = \left(\frac{1}{2} - \frac{1}{2^{2p}}, \frac{1}{2} + \frac{1}{2^{2p}}\right)$. On the other hand, consider the other iterates $D^n(\varphi_H(\theta))$ for $n \geq 1$. If $n < 2p-1$, then $D^n(\varphi_H(\theta))$ begins with 11 so it does not lie in U_p. For $n = 2p-1$, then $D^n(\varphi_H(\theta)) = 1\cdot\theta$ also does not belong to U_p, as θ cannot begin with $2p-1$ zeros by the above Lemma. Finally, if $n \geq 2p$, then

$$D^n(\varphi_H(\theta)) = D^{n-2p}(\theta) \notin U_p$$

again by the Lemma. In conclusion, since $\varphi_H(\theta)$ belongs to U_p and none of its forward iterates does, then $\varphi_H(\theta)$ is closer to $1/2$ than all its iterates, hence $\varphi_H(\theta)$ belongs to $\mathcal{R}$. □

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