Fundamental Openness Principle and Zariski's Main Theorem - Definitions and reminders

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Definitions

Morphism (regular map): $\varphi : X \to Y$ morphism of affines if in local coordinates $\{x_i\}$ and $\{y_i\}$, image of $(x_1, \cdots, x_n)$ given by

$$\forall i = 1, \cdots, m\text{: } y_i = \varphi_i(x_{1,\dots,} x_n), \text{ w/} \varphi_i \in \mathbb{C}(X).$$

Morphism φ is **dominant** when $\overline{\varphi(X)} = Y$ in Zariski topology, and is **smooth** at $x \notin \text{Sing}X$ if $\varphi(x) \notin \text{Sing}Y$, $(d\varphi)_x$ surjective.

Dominant φ **birational** if $\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ is isomorphism.

Top. Unibranch: X is topologically unibranch at a point x if:

$\forall \mathcal{U}_x$cl-open (classically) nhd. and $\forall Y \subsetneq X$ alg. closed, $\exists\ \mathcal{V}_x \subset \mathcal{U}_x$

cl-open nhd. of x s.th. $\mathcal{V}_x \setminus Y$ is cl-connected, e.g. if $x \notin \mathrm{Sing}X$.

Notations: For $X \subset \mathbb{C}^n$ variety and ideal $I \subset \mathbb{C}[X]$ we write

$V_X(I) := V(I) \cap X$. The prefix "cl-" means in classical topology.

Reminders of basic facts

F1 [M1.14]: $Y \subsetneq X$ proper subvariety $\Longrightarrow \dim Y < \dim X$.

F2 [M2.33]: $X^{(r)} \subset \mathbb{P}^n$ and $\mathcal{U} \subset X$ Zar-open then $\overline{\mathcal{U}}^{cl} = X$.

F3 [M3.11]: $f : X \to Y$ continuous map of loc. cmpct spaces. If $y \in Y$ s.t. $f^{-1}(y)$ cmpct, then: $\exists$ open nhds. $\mathcal{U} \supset f^{-1}(y)$ and $\mathcal{V} \supset f(\mathcal{U})$ s.t. $f_{|\mathcal{U}} : \mathcal{U} \to \mathcal{V}$ proper.