Fundamental Openness Principle and Zariski's Main Theorem

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Main Results

Below: $\varphi : X^{(r)} \to Y^{(s)}$ regular dominant map of affine varieties,

$r = \dim X$, $s = \dim Y$ and $X \subset \mathbb{C}^n$, $Y \subset \mathbb{C}^m$.

I - Fundamental Openness Principle (FOP):

Suppose $r = s$, take $x \in X$ s.t. Y topologically unibranch at $\varphi(x)$

and $\{x\} \subset \varphi^{-1}(\varphi(x))$ component, then φ open at x.

II - Dimension of Fibres Theorem (FibDim):

For all $y \in Y$ and any $W \subset \varphi^{-1}(y)$ component, $\dim W \geq r - s$.

III - Zariski's Main Theorem - Affine Smooth (ZMT):

Suppose $\varphi : X^{(r)} \to Y^{(r)}$ birational. Fix $a \in X$ with

$b = \varphi(a) \in Y \setminus \operatorname{Sing} Y$. Two possibilities:

(1) Either $\exists\ \varphi^{-1} : \mathcal{U}_b \to X$ regular on Zariski-open nhd. $\mathcal{U}_b \subset Y$;

(2) Or, $\exists$ subvariety $E \subset X$, $a \in E$ such that $\dim E = r - 1$

and $\dim \overline{\varphi(E)} \leq r - 2$ (in particular, $\varphi^{-1}(b)$ has component

through a of dim.> 0).

I - Fundamental Openness Principle

Lemma I.1: For $\varphi : X \to Y$ regular, $\overline{\varphi(X)} = V_Y(\ker \varphi^*)$. Thus φ dominant iff $\varphi^* : \mathbb{C}(X) \to \mathbb{C}(Y)$ injective.

Pf: $\ker \varphi^* \subset I(\varphi(X)) \implies \overline{\varphi(X)} = V_Y(I(\varphi(X))) \subset V_Y(\ker \varphi^*)$.

Conversely, if $V_Y(J)$ Zar-closed contains $\varphi(X)$, then $\sqrt{J} \subset \sqrt{\ker \varphi^*}$ necessarily, so $V_Y(\ker \varphi^*) \subset V_Y(\sqrt{J}) = V_Y(J)$. If true for any Zar-closed$\supset \varphi(X)$, then $V_Y(\ker \varphi^*) \subset \overline{\varphi(X)}$. Second claim follows from $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ injective iff $\overline{\varphi(X)} = V_Y(0) = Y$. □

I - Fundamental Openness Principle

Lemma I.2: Smooth points of φ form $\neq \emptyset$ Zar-open subset of X.

Pf: Since $\varphi : X^{(r)} \to Y^{(s)}$ is dominant, $\varphi(X) \setminus \mathrm{Sing}\, Y \neq \emptyset$ so

$X \setminus \varphi^{-1}(\mathrm{Sing}\, Y) \neq \emptyset$, and $X_0 := X \setminus (\mathrm{Sing} X \cup \varphi^{-1}(\mathrm{Sing}\, Y)) \neq \emptyset$.

Set $\mathrm{Cr}\varphi := \{x \in X_0 : \ \mathrm{rk}(d\varphi)_x < s\} \cup (X \setminus X_0)$ is closed analytic,

but is also algebraic (via local alg. coords) and $\dim \mathrm{Cr}\varphi < r$.

Moreover: $\mathrm{CrV}(\varphi) := \overline{\varphi(\mathrm{Cr}\varphi)} \subsetneq Y^{(s)}$ by Sard and $\dim \mathrm{CrV}(\varphi) < s$.

□

I - Fundamental Openness Principle

Pf. FOP: Y top. unibranch at $b = \varphi(a)$ and $\{a\} \subset \varphi^{-1}(b)$ comp.

1) Choose cl-open nhds $\mathcal{V}_b \subset Y$, $\{a\} \subset \mathcal{U}_a \subset X$. Since φ regular,

$\varphi_{|\mathcal{U}_a} \in \mathcal{C}^0(\mathcal{U}_a)$, shrinking these nhds, $\varphi_{|\mathcal{U}_a} : \mathcal{U}_a \to \mathcal{V}_b$ is proper.

2) (Y, b) unibranch: $\exists\, \mathcal{V}'_b \subset \mathcal{V}_b$ s.th. $\mathcal{V}'_b \setminus \mathrm{CrV}(\varphi)$ connected.

Let $\mathcal{U}'_a := \mathcal{U}_a \cap \varphi^{-1}(\mathcal{V}'_b)$. We'll show $\varphi(\mathcal{U}'_a) = \mathcal{V}'_b$. Indeed, with

$B := \mathrm{CrV}(\varphi)$ map $\psi := \varphi_{|\mathcal{U}'_a \setminus \varphi^{-1}(B)}$ is a local homeo., smooth and

proper. So $y \mapsto \#\psi^{-1}(y)$ is locally const., thus constant and > 0.

Finally $\varphi(\mathcal{U}'_a)$ closed in $\mathcal{V}'_b$, since $\varphi_{|\mathcal{U}'_a}$ is proper and images of closed sets under proper maps are closed. Done. $\square$

II - Dimension of Fibres

Lemma II.1: Let $\varphi : X \to \mathbb{C}^k$ morphism of affines s.th. $\{a\} \subset \mathbb{C}^n$ is a component of $\varphi^{-1}(0)$, and k is the smallest dimension for which there exists such a map. Then φ is dominant.

Pf: For φ not dominant $l := \dim \overline{\varphi(X)} < k$. Let $h_i \in \mathfrak{m}_{\overline{\varphi(X)},0}$ for $1 \leq i \leq l$, s.th. $\forall$ i: $h_i \not\equiv 0$ on any comp. of $V_{\overline{\varphi(X)}}(h_{i-1})$. Then $\psi : X \ni x \mapsto (\varphi^* h_1, \cdots, \varphi^* h_l)(x) \in \mathbb{C}^l$ is s.th. $\{a\} \subset \varphi^{-1}(0)$ is a comp. since $\dim V_{\overline{\varphi(X)}}(\{h_i\}_{i=1}^l) = 0$, so k wasn't minimal. $\square$

Pf FibDim: Say $b \in Y$, $W \subset \varphi^{-1}(b)$ component s.th. $r - s >$ $\dim W$, and $a \in W \setminus \{\text{other } \varphi^{-1}(b) \text{ comp}\}$. We'll construct dominant regular $\Phi : X \to \mathbb{C}^r$ to get contradiction with FOP:

1) <u>$\exists\, \rho : X \to \mathbb{C}^{r-1}$ **reg. s.th.** $\{a\}$ **component of** $\rho^{-1}(0)$**:**</u>

$\forall\, g_1 \in \mathfrak{m}_{Y,b} \setminus I(Y)$, $\dim V_Y(g_1) < s$ etc., i.e. $\exists\, \{g_i\}_{i=1}^{s} \subset \mathfrak{m}_{Y,b}$ s.th. $\{b\}$ comp. of $V_Y(g_1, \cdots, g_s)$. Likewise $\exists\, \{f_i\}_{i=1}^{r-s-1} \subset \mathfrak{m}_{X,a}$ s.th.$\{a\}$ comp. of $V_X(f_1, \cdots, f_{r-s-1})$. Then our $\rho : X \to \mathbb{C}^{r-1}$ is $x \mapsto (f_1, \cdots, f_{r-s-1}; \varphi^* g_1, \cdots, \varphi^* g_s)(x)$.

2) Reducing "$r-1$" above to a minimal k:

If $k = \dim \overline{\rho(X)} < r-1$, take $\{h_i\}_{i=1}^k \subset \mathbb{C}[\overline{\rho(X)}]$ s.th. $\{0\}$ is a component of $V_{\overline{\rho(X)}}(\{h_i\})$. Let $H := (h_1, \cdots h_k)$ and $\psi := H \circ \rho$.

Now say k is minimal in the sense of [II.1] s.th. $\exists\, \psi : X \to \mathbb{C}^k$ with $\{a\}$ comp. of $\psi^{-1}(0)$. Then $\overline{\psi(X)} = \mathbb{C}^k$. Say $\{z_i\}$ coords of $\mathbb{C}^k$.

3) <u>**Construction of $\Phi : X \to \mathbb{C}^r$:**</u> Take $\psi : X \to \mathbb{C}^k$ from 2).

Complete $\{\psi^* z_i\}_{i \leq k}$ to transcendence basis of $\mathbb{C}(X)$ over $\mathbb{C}$ with $\{w_i\} \subset \mathbb{C}[X]$ s.th. $w_i(a) = 0\ \forall i$. Then our $\Phi : X \to \mathbb{C}^r$ maps $x \mapsto (\psi(x), w_1(x), \cdots, w_{r-k}(x))$, and is dominant.

4) Completion of proof by a contradiction:

Morphism Φ dominant, $\mathbb{C}^r$ is top. unibranch at 0, and $\{a\}$ is a comp. of $\Phi^{-1}(0)$. By FOP, $\exists\, \mathcal{U}_a$ cl-open nhd s.t. $\Phi(\mathcal{U}_a)$ cl-open.

On the other hand: if $x \in \mathcal{U}_a$ s.th. $\Phi(x) = (0, w_1(x), \cdots w_{r-k}(x))$, then $x \in \psi^{-1}(0) \cap \mathcal{U}_a = \{a\} \Longrightarrow \Phi(\mathcal{U}_a)$ isn't cl-open ?! Done. $\square$

III - Zariski's main theorem

Pf ZMT: $\varphi : X \to Y$ reg. birational, by def. $\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X)$ is isomorphism, and locally

$$x_i = \varphi^* \left(\frac{a_i(y_1, \cdots, y_r)}{b_i(y_1, \cdots, y_r)} \right) \text{ and } a_i, b_i \in \mathbb{C}[Y], \, \forall\, 1 \leq i \leq r,$$

where the $b_i \not\equiv 0$ in nhd of $\varphi(x) = y$. Choose $b \in Y \setminus \operatorname{Sing} Y$, $\mathcal{O}_{Y,b}$ UFD, so $gcd(a_i, b_i) = 1$. Let $\theta = \prod_i b_i$.

<u>If $\theta(b) \neq 0$:</u> Then exists local inverse to $\varphi : X \to Y$, namely

$$\psi : Y \setminus V_Y(\theta) \ni (y_1, \cdots, y_r) \mapsto \left(\frac{a_1}{b_1}(y), \cdots, \frac{a_r}{b_r}(y) \right) \in X$$

<u>If $\theta(b) = 0$:</u> Say $b_1(b) = 0$ and $\beta \in \mathbb{C}[Y]$ irreducible factor of $b_1 \in \mathcal{O}_{Y,b}$ s.th. $\beta(b) = 0$. Take $E \subset X \cap V(\varphi^*\beta)$ irred. comp. at a. Then $\dim E = r - 1$. We claim $\dim \overline{\varphi(E)} \leq r - 2$. Indeed, $\varphi^*(a_1) = x_1 \cdot \varphi^*(\beta) \cdot \varphi^*(b_1')$, and so $\forall\, x \in E$ holds $(\varphi^* a_1)(x) = 0$. Thus $a_1 = \beta = 0$ on $\overline{\varphi(E)}$ and $a_1 \notin \beta \cdot \mathcal{O}_{Y,b} \in \mathrm{Spec}\mathcal{O}_{Y,b}$. So, $a_1 \notin \mathfrak{P} := \left\{ f \in \mathbb{C}[Y] \big/ f \in \beta \cdot \mathcal{O}_{Y,b} \right\} \in \mathrm{Spec}\mathbb{C}[Y]$. Therefore $\overline{\varphi(E)} \subset V(a_1) \cap V(\mathfrak{P}) \subsetneq V(\mathfrak{P}) \subsetneq Y$ implying $\dim \overline{\varphi(E)} \leq r - 2$.

□