(1) (a) Let G be a Lie group with a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. Prove that it's biinvariant if and only if its restriction to $\mathfrak{g} = T_e G$ is invariant under Ad_g for any $g \in G$.

(b) Let N^3 be the 3-dimensional Heisenberg group

$$N = \{A = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid \text{where } x, y, z \in \mathbb{R}\}$$

Prove that N does not admit a bi-invariant Riemannian metric.

(2) (a) Verify that the left invariant volume form on the 3-dimensional Heisenberg group N^3 is bi-invariant.

(b) Verify that the left invariant volume form on $\mathrm{GL}(2, \mathbb{R})$ is NOT right invariant.

(c) Prove that $\mathrm{SU}(2)$ is isomorphic to $\mathrm{Sp}(1)$.
Hint: Indentify $\mathbb{H}$ with $\mathbb{C}^2$ and view elements of $\mathrm{Sp}(1)$ as complex linear maps $\mathbb{C}^2 \to \mathbb{C}^2$ which preserve the norm and commute with left multiplicition by j.