MAT 1300 **Term Test Solutions** **October 21, 2014**

(1) (8 pts) Give the following definitions

(a) A tangent vector to a smooth manifold

(b) A smooth manifold with boundary.

Solution

(a) A tangent vector to a smooth manifold M at a point p is a map $v\colon C^\infty(M) \to \mathbb{R}$ satisfying the following conditions

(i) v is linear, i.e. $v(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 v(f_1) + \lambda_2 v(f_2)$ for any $\lambda_1, \lambda_2 \in \mathbb{R}$ and $f_1, f_2 \in C^\infty(M)$.

(ii) $v(f \cdot g) = v(f)g(p) + f(p)v(g)$ for any $f, g \in C^\infty(M)$.

(b) A smooth manifold with boundary is a generalized smooth manifold with boundary which is Hausdorff and admits a countable atlas. A generalized smooth manifold with boundary is a set M with a collection of maps $\{\psi_\alpha\colon V_\alpha \to U_\alpha\}_{\alpha\in\mathcal{A}}$ where $V_\alpha \subset \mathbb{H}^n$, $U_\alpha \subset M$, such that the following conditions are satisfied

(i) $\cup_\alpha U_\alpha = M$;

(ii) $\psi_\alpha\colon V_\alpha \to U_\alpha$ is 1-1 and onto;

(iii) $V_{\alpha\beta} = \psi_\alpha^{-1}(U_\beta)$ is open in $\mathbb{H}^n$ for any α, β.

(iv) $\psi_\beta^{-1} \circ \psi_\alpha\colon V_{\alpha\beta} \to V_{\beta\alpha}$ is smooth for any $\alpha, \beta \in \mathcal{A}$.

(2) (10 pts) Let $f\colon \mathbb{RP}^n \to \mathbb{R}$ be given by $f([x_0 : x_1 : \ldots : x_n]) = \frac{x_n^2}{x_0^2+x_1^2+\ldots+x_n^2}$.

(a) Show that f is well defined and smooth.

(b) Show that the set $\{f = 1/2\}$ is nonempty and carries a natural structure of a manifold of dimension $n-1$.

Solution

(a) Let $\tilde{f}\colon \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}$ be given by $\tilde{f}(x_0, x_1, \ldots, x_n]) = \frac{x_n^2}{x_0^2+x_1^2+\ldots+x_n^2}$. Then for any $\lambda \neq 0$ and any $x \in \mathbb{R}^{n+1}\backslash\{0\}$ we have $\tilde{f}(\lambda\cdot x) = \frac{\lambda^2\cdot x_n^2}{\lambda^2\cdot x_0^2+\ldots+\lambda^2\cdot x_n^2} = \frac{x_n^2}{x_0^2+\ldots+x_n^2} = \tilde{f}(x)$ which means that f is well defined. To see that f is smooth consider its representation in standard coordinate charts $\phi_i\colon \mathbb{R}^n \to \mathbb{RP}^n$ $(i = 0, \ldots, n))$ given by $\phi_i(x_1, \ldots, x_n) = [x_1 : \ldots : x_i : 1 : x_{i+1} : \ldots : x_n]$.

For $i < n$ we have $f \circ \phi_i(x_1, \ldots, x_n) = \frac{x_n^2}{x_1^2+\ldots+1+\ldots+x_n^2}$ is smooth in x. And for $i = n$ we have $f \circ \phi_n(x_1, \ldots, x_n) = \frac{1}{x_1^2+\ldots+x_n^2+1}$ is also smooth in x.

Thus f is smooth.

(b) $f(1 : 0 : \ldots : 0 : 1) = 1/2$ and hence $\{f = 1/2\}$ is nonempty. To see that $\{f = 1/2\}$ carries a natural structure of a manifold of dimension $n-1$ it's sufficient to check that $1/2$ is a regular value of f. Since the target is 1-dimensional it's enough to show that $df_p \neq 0$ for any $p \in \{f = 1/2\}$.

We'll check it for points lying in $U_i = \phi_i(\mathbb{R}^n)$ for all $i = 0, \ldots, n$.

For $i < n$ we have $g_i(x) = f \circ \phi_i(x_1, \dots, x_n) = \frac{x_n^2}{x_1^2+\ldots+1+\ldots+x_n^2}$.
We compute $\frac{\partial g_i}{\partial x_n}(x) = \frac{2x_n(1+x_1^2+\ldots x_{n-1}^2)}{(x_1^2+\ldots+1+\ldots+x_n^2)^2} \neq 0$ unless $x_n = 0$. However, if $x_n = 0$ then $g_i(x) = 0 \neq 1/2$ and hence $dg_i(x) \neq 0$ for any x satisfying $g_i(x) = 1/2$.
Similarly, for $i = n$ we have $g_n(x) = \frac{1}{x_1^2+\ldots+x_n^2+1}$. We compute

$$\frac{\partial g_i}{\partial x_i}(x) = \frac{-2x_i}{x_1^2 + \ldots + x_n^2 + 1}$$

Thus $dg_n(x) = 0$ only if all $x_i = 0$, i.e. $x = 0$. However, $g_n(0) = \frac{1}{1} = 1 \neq 1/2$ and hense we also have $dg_n(x) \neq 0$ for any x satisfying $g_n(x) = 1/2$.
Altogether this means that $df_p \neq 0$ for any p satisfying $f(p) = 1/2$ i.e. $1/2$ is a regular value of f.

(3) (12 pts) Mark **True or False**. You DO NOT need to justify your answers.
Let M, N be smooth manifolds.

(a) A submersion $f\colon M \to N$ is onto.
Answer: **False**. E.g. take the inclusion $i\colon (-1,1) \to \mathbb{R}$.

(b) A smooth embedding $f\colon M \to N$ is a closed map.
Answer: **False**. Same example as in (a).

(c) If a smooth map $f\colon M \to N$ is injective then it's an immersion.
Answer: **False**. E.g. $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

(d) A local diffeomorphism which is 1-1 and onto is a diffeomorphism.
Answer: **True**.

(e) $\mathbb{S}^1$ is diffeomorphic to $\mathbb{RP}^1$.
Answer: **True**.

(f) Composition of two maps of constant rank has constant rank.
Answer: **False**. E.g. take $f\colon (-1,1) \to R^2$ given by $f(x) = (x, \sqrt{1-x^2})$ and $g\colon \mathbb{R}^2 \to \mathbb{R}$ given by $g(x,y) = y$.
Both maps have constant rank 1. However, $h(x) = g \circ h(x) = \sqrt{1-x^2}$ does not have constant rank.

(4) (10 pts) Let M^n, N^m be smooth manifolds such that $n > m$. Let $f\colon M^n \to N^m$ be a smooth map.
Prove that f is not $1-1$.
Hint: Use the constant rank theorem.

Solution

Let $p \in M$ be a point where $r = \operatorname{rank} df_p$ is maximal possible. Obviously, $r \leq m < n$. By reducing domain and range to co-ordinate charts we can assume that $M = U$ is an open subset in $\mathbb{R}^n$ and $N = \mathbb{R}^m$. By permuting co-ordinates we can assume that the matrix $[\frac{\partial f_i}{\partial x_j}(p)]_{i,j=1,\ldots r}$ is invertible.

By continuity of determinants the same holds true for points x near p. Thus, df_x has rank $\geq r$ for x near p. But since r was chosen to be maximal possible we actually have $\operatorname{rank} df_x = r$ for x near p.

Thus, by the constant rank theorem after a change of coordinates near p on the domain and the target we can assume that f has the form $f(y_1, \dots y_n) = (y_1, \dots, y_r, 0, \dots, 0)$ which is not 1-1 since $r < n$. $\square$

(5) (10 pts)
   (a) Let $f\colon M \to \mathbb{R}$ be smooth and let V be a smooth vector field on M. Suppose $V(p)(f) > 0$ for some p in M.
   Prove that there is an open set U containing p such that $V(x)(f) > 0$ for any $x \in U$.
   (b) Let $f\colon M \to \mathbb{R}$ be smooth. Suppose for every $p \in M$ there exists $v \in T_pM$ such that $v(f) > 0$.
   Prove that there exists a smooth vector field V on M such that $V(p)(f) > 0$ for every $p \in M$.
   Hint: Use partition of unity.

Solution

(a) In some local coordinates V has the form $V(x) = \Sigma_{i=1}^n v_i(x) \frac{\partial}{\partial x_i}|_x$ where $v_i(x)$ are smooth functions.
Then $g(x) = V(x)(f) = \Sigma_{i=1}^n v_i(x) \frac{\partial f}{\partial x_i}(x)$ is smooth in x. In particular, it's continuous and since $g(p) > 0$, by continuity, there exists an open set U containing p such that $g(x) > 0$ for any $x \in U$. $\square$

(b) Let $p \in M$ be any. We are given that there exists $v \in T_pM$ such that $v(f) > 0$. In some local coordinates x it has the form $v = \Sigma_{i=1}^n v_i \frac{\partial}{\partial x_i}|_p$ for some $v_1, \dots v_n \in \mathbb{R}$. Extend v to a smooth vector field V_p on a small open set U_p containing p by the formula $V_p(x) = \Sigma_{i=1}^n v_i \frac{\partial}{\partial x_i}|_x$. By part (a) we have that $V(x)(f) > 0$ on some open set $W_p \subset U_p$ containing p.
The collection $\{W_p\}_{p \in M}$ is an open cover of M. Let $\{\phi_i\}_{i=1}^\infty$ be a partition of unity subordinate to this cover so that $\operatorname{supp} \phi_i \subset W_{p_i}$.
Let $V = \sum_i \phi_i \cdot V_{p_i}$. We claim that V satisfies the required properties: V is obviously smooth and for any $p \in M$ we have

$$V(p)(f) = \sum_i \phi_i \cdot V_{p_i}(p)(f) \geq 0 \tag{1}$$

since all the terms in the sum are nonnegative. Moreover, for every p there is an i such that $\phi_i(p) > 0$. For that i we have that $p \in \operatorname{supp} \phi_i \subset W_{p_i}$ and therefore, $V_{p_i}(p)(f) > 0$. Thus, at least one term in the sum (1) is positive and hence, $V(p)(f) > 0$. $\square$