(1) Let $O(n) = \{A \in M(n \times n)|$ such that $A \cdot A^t = Id$. Prove that $O(n)$ is a smooth manifold.

Hint. Consider the map $F\colon M(n \times n) \to M(n \times n)$ given by $F(A) = A \cdot A^t$. Then $O(n) = \{F = Id\}$. Notice that $F(A)$ is always symmetric.

(2) Recall that given a topological space X and a subset $A \subset X$ the induced topology on A is defined as follows. A set $U \subset A$ is called open in A if $U = W \cap A$ for some set $W \subset X$ open in X.

Let M^n be a manifold with boundary. Consider the induced atlas on the boundary ∂M giving ∂M the structure of a smooth manifold. prove that the topology defined on ∂M is the same as the induced topology from M.

(3) Recall that given a topological space X and an equivalence relation $\sim$ on X we can endow the quotient space $X/\sim$ with topology as follows.

Let $\pi\colon X \to X/\sim$ be the canonical projection map. Define a set $U \subset X/\sim$ to be open iff $\pi^{-1}(U)$ is open in X. This topology is called the *quotient topology* on $X/\sim$.

Recall that $\mathbb{RP}^n$ is the set of lines through the origin in $\mathbb{R}^{n+1}$. Each such line intersects the unit sphere S^n in a pair of antipodal points. Thus as a set $\mathbb{RP}^n$ can be identified with $S^n/\sim$ where $x, y \in S^n$ are equivalent iff $x = \pm y$.

Prove that the topology on $\mathbb{RP}^n$ given by the standard smooth structure is equal to the quotient topology induced by $\sim$ from the standard topology on S^n.