(1) A manifold M is called *path connected* if any two points $p, q \in M$ can be connected by a curve in M, i.e. if there exists a continuous map $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p, \gamma(1) = q$.

Prove that a manifold M is connected if and only if it's path connected.

(2) Let M, N_1, N_2 be manifolds (possibly with boundary). Let $f = (f_1, f_2)\colon M \to N_1 \times N_2$ where $f_i\colon M \to N_i$. Prove that f is smooth if and only if both f_1 and f_2 are smooth.

(3) Let $M = \mathbb{R}^2/\sim$ where $(x_1, y_1) \sim (x_2, y_2)$ iff $x_1 - x_2$ and $y_1 - y_2$ are integers. Let $\pi\colon \mathbb{R}^2 \to M$ be the canonical projection map.

Consider the following collection of local parameterizatioon $\psi_{a,b,c,d}\colon (a,b) \times (c,d) \to M$ given by $\psi_{a,b,c,d} = \pi|_{(a,b)\times(c,d)}$ where $a, b, c, d \in \mathbb{R}, 0 < b - a < 1, 0 < d - c < 1$.

Prove that this collection of maps gives M^2 a structure of a smooth 2-dimensional manifold and that M is diffeomorphic to $S^1 \times S^1$.

Hint: In this problem as well as in problem (4) you can use the following general observation (prove it first!) which immediately follows from the Inverse Function Theorem:

Suppose $U \subset \mathbb{R}^n$ is open and $f\colon U \to \mathbb{R}^n$ is smooth, 1-1 and the matrix of partial derivatives $[\frac{\partial f_i}{\partial x_j}(p)]$ is invertible for every $p \in U$.

Then $V = f(U)$ is open in $\mathbb{R}^n$ and $f\colon U \to V$ is a diffeomorphism.

(4) Look at the surface of revolution M^2 in $\mathbb{R}^3$ obtained by rotating the circle of radius 1 centered at $(2, 0)$ around the vertical axes.

(a) Verify that it's given by the equation

$$(\sqrt{x^2 + y^2} - 2)^2 + z^2 = 1$$

and that M is a smooth manifold.

(b) Prove that M is diffeomorphic to $S^1 \times S^1$.

(5) Let $f\colon M \to N$ be a diffeomorphism between manifolds with boundary.

Prove that $f(\partial M) = \partial N$ and $f|_{\partial M}\colon \partial M \to \partial N$ is a diffeomorphism.