(1) Let V be a vector field on M i.e. it's a map $V\colon M \to TM$ such that $V(p) \in T_pM$ for every $p \in M$. Then for any smooth function $f\colon M \to \mathbb{R}$ we have a new function $V(f)\colon M \to \mathbb{R}$ given by $V(f)(p) = V(p)(f)$ (recall that $V(p)$ is a derivation at p).

Prove that V is smooth if and only if $V(f)$ is smooth for any smooth f.

(2) Let $M = O(n) = \{A \in M(n \times n)|$ such that $A \cdot A^t = Id\}$ with its canonical smooth structure defined in homework 1.

(a) Show that the multiplication map $F\colon M \times M \to M$ given by $F(A,B) = A \cdot B$ is a smooth map. Conclude that for any $A \in O(n)$ the left translation map $l_A\colon M \to M$ given by $l_A(B) = A \cdot B$ is a diffeomorphism.

(b) A vector field V on $O(n)$ is called *left invariant* if for any $A, B \in M$ we have that $dl_A(V(B)) = V(l_A(B))$.
Let $e \in O(n)$ be the identity matrix. Prove that for any $v \in T_eO(n)$ there is a unique left invariant vector field V on $O(n)$ such that $V(e) = v$.

(c) Prove that any left invariant vector field on $O(n)$ is smooth. *Hint: use problem (1)*

(d) Prove that $TO(n)$ is diffeomorphic to $O(n) \times \mathbb{R}^{n(n-1)/2}$.

(3) Let M^n be a manifold without boundary. Let $p \in M$ and $v \in T_pM$. Prove that there exists a smooth curve $\gamma\colon (-1,1) \to M$ such that $\gamma(0) = p$ and $\gamma'(0) = v$.

(4) Let V be a smooth vector field on M. Prove that $V\colon M \to TM$ is an immersion.

(5) Let $U \subset \mathbb{R}^{n+k}$ be an open subset and let $F\colon U \to \mathbb{R}^k$ be a smooth map. Let $c \in \mathbb{R}^k$ be a regular value of F and let $M = \{F = c\}$ be the level set with its canonical smooth structure.

Prove that the inclusion $i\colon M \to \mathbb{R}^{n+k}$ is an immersion.