(1) Prove that $\mathbb{S}^1$ is diffeomorphic to $\mathbb{RP}^1$.

Hint: Consider the map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ given by $f(z) = z^2$ and verify that it induces a diffeomorphism $\hat{f}\colon \mathbb{RP}^1 \to \mathbb{S}^1$.

(2) (a) Let $S \subset M^n$ be a k-dimensional submanifold in a smooth manifold M^n. Let $p \in S$ be any point.
Prove that there exists an open set $U \subset M$ and a smooth map $\Phi\colon U \to \mathbb{R}^{n-k}$ such that 0 is a regular value of Φ and $S \cap U = \Phi^{-1}(0)$.

(b) Let $S \subset \mathbb{R}^n$ be a k-dimensional submanifold. Let $p \in S$. Show that there an open set $U \subset \mathbb{R}^n$ containing p such that up to reordering of coordinates on $\mathbb{R}^n$ $U \cap S$ is equal to the graph of a smooth function $f\colon W \to \mathbb{R}^{n-k}$ where $W \subset \mathbb{R}^k$ is open.
Hint: use part a)

(c) Let $S \subset \mathbb{R}^3$ be the graph of $z = \sqrt[4]{x^2 + y^2}$.
Prove that S is not a smooth submanifold of $\mathbb{R}^3$.
Hint: use part b)