(1) Let $e_1, \dots e_n$ be a basis of a vector space V. Recall that for a multi-index $I = (i_1, \dots i_k)$ we define e^I by the formula $e^I(v_1, \dots v_k) = \det A$ where A is the k by k matrix formed by the rows $i_1, \dots i_k$ of the matrix $[v_1|v_2|\dots v_n]$ when v_i's are written with respect to the basis $e_1, \dots e_n$.

Prove that $e^{1,2,\dots n} = n! \cdot Alt(e^1 \otimes e^2 \otimes \dots \otimes e^n)$.

(2) Let $\sigma \in S_{k+l}$ be the following permutation

$$\sigma = \begin{pmatrix} 1 & \cdots & k & k+1 & \cdots & k+l \\ l+1 & \cdots & l+k & 1 & \cdots & l \end{pmatrix}$$

Show that $\text{sign}(\sigma) = (-1)^{kl}$

(3) Let V be a finite dimensional vector space and let $T \in \mathcal{T}^k(V), S \in \mathcal{T}^l(V)$. Suppose $Alt(T) = 0$.

Prove that $Alt(T \otimes S) = 0$.

(4) let V be a vector space and let $e_1, \dots, e_n$ be a basis of V. Let $\{e^I\}$ be the corresponding standard basis of $\mathcal{A}^k(V)$ - the space of all alternating k-tensors on V. Let $L\colon V \to V$ be a linear map and let $A = [L]$ be the matrix of L with respect to the basis $e_1, \dots, e_n$.

Let $L^*\colon \mathcal{A}^k(V) \to \mathcal{A}^k(V)$ be the linear map induced by A.

Find the matrix of A^* with respect to the basis $\{e^I\}$, i.e. find the coefficients λ_J in the expansion $A^*(e^I) = \sum_J \lambda_J e^J$.

(5) Let V be a finite dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. Let $e_1, \dots e_n$ be on orthonormal basis of V.

Define an inner product on $\mathcal{T}^k(V)$ by declaring the standard basis $\{e^*_{i_1} \otimes \dots \otimes e^*_{i_k}\}$ to be orthonormal.

Prove that this inner product does not depend on the choice of the orthonormal basis $e_1, \dots e_n$.

Hint: First prove the statement for $k = 1$ and then show that for decomposable tensors of the form $T = \omega_1 \otimes \dots \otimes \omega_k$ and $S = \eta_1 \otimes \dots \otimes \eta_k$ with $\omega_i, \eta_i \in V^*$ we must have $\langle T, S \rangle = \prod_{i=1}^k \langle \omega_i, \eta_i \rangle$