Definition 0.0.1. *A smooth atlas on X is called maximal if it is not contained in any other atlas.*

Two atlases $\mathcal{A} = \{\psi_\alpha \colon V_\alpha \to U_\alpha\}_{\alpha \in A}$, and $\mathcal{A}' = \{\psi_{\alpha'} \colon V_{\alpha'} \to U_{\alpha'}\}_{\alpha \in A'}$ on X are compatible if their union is still an atlas.

$\mathcal{A}$ and $\mathcal{A}'$ are said to define the same smooth structure on X if they are compatible.

Example 0.0.2. Let $M = \mathbb{R}$ and let $\mathcal{A}_1$, $\mathcal{A}_2$ be two atlases on M each consisting of a single map $\mathcal{A}_1 = \{\psi_1\}, \mathcal{A}_2 = \{\psi_2\}$ where $\psi_i \colon \mathbb{R} \to \mathbb{R}$ (i=1,2) are given by $\psi_1(x) = x, \psi_2(x) = x^3$. These atlases are not compatible because $\psi_2^{-1} \circ \psi_1(x) = \sqrt[3]{x}$ is not smooth at 0.

Theorem 0.0.3. *Let X be a set that admits a smooth structure.*

1. *Any atlas $\mathcal{A}$ on X is contained in a unique maximal atlas $\mathcal{A}_{max}$*
2. *Two atlases $\mathcal{A}$ and $\mathcal{A}'$ are compatible iff their maximal atlases $\mathcal{A}_{max}$ and $\mathcal{A}'_{max}$ are equal.*

   Therefore, two atlases define the same smooth structure iff their maximal atlases are equal.

Proof. (1) Let $\mathcal{A}$ on X. Let $V \subset \mathbb{R}^n$ be open. We will say that a map $\psi \colon V \to X$ is compatible with $\mathcal{A}$ if $\mathcal{A} \cup \psi$ is still an atlas, i.e. if the following conditions hold

- $\psi \colon V \to U = \psi(V)$ is a bijection.
- The sets $\psi^{-1}(U_\alpha)$ and $\psi_\alpha^{-1}(U)$ are open in $\mathbb{R}^n$ for any α.
- The transition maps $\psi^{-1} \circ \psi_\alpha$ and $\psi_\alpha^{-1} \circ \psi$ are smooth.

Let $\mathcal{A}_{max}$ be the union of all maps compatible with $\mathcal{A}$. It's obvious that $\mathcal{A} \subset \mathcal{A}_{max}$ and that any atlas compatible with $\mathcal{A}$ is contained in $\mathcal{A}_{max}$.

We claim that $\mathcal{A}_{max}$ is an atlas. To see this we need to verify that for any two maps in $\mathcal{A}_{max}$ $\psi' \colon V' \to U'$ and $\psi'' \colon V'' \to U''$ we have

- $W' = \psi'^{-1}(U'')$ and $W'' = \psi''^{-1}(U')$ are open in $\mathbb{R}^n$ (Exercise)
- The transition map $\psi''^{-1} \circ \psi' \colon W' \to W''$ is smooth. To see this let $p \in U' \cap U''$. Then $p = \psi'(p') = \psi''(p'')$ for some $p' \in W', p'' \in W''$. Then $p \in U\alpha$ for some α.
  Therefore near p' we can rewrite $\psi''^{-1} \circ \psi'$ as $\psi''^{-1} \circ \psi' = (\psi''^{-1} \circ \psi_\alpha) \circ (\psi_\alpha^{-1} \circ \psi')$. Since both $\psi''^{-1} \circ \psi_\alpha$ and $\psi_\alpha^{-1} \circ \psi'$ are smooth where defined we conclude that $\psi''^{-1} \circ \psi'$ is smooth near p' as a composition of two smooth maps.

This proves that $\mathcal{A}_{max}$ is an atlas and hence it's the unique maximal atlas containing $\mathcal{A}$.

(2) Homework.

□

Definition 0.0.4. *Let $U \subset \mathbb{R}^n$ be open and let $F \colon \mathbb{R}^n \to \mathbb{R}^k$ be a smooth map. A point $c \in \mathbb{R}^k$ is called a regular value of F if for any point p on the* *level set $\{F = c\}$ we have that the differential $dF_p\colon \mathbb{R}^n \to \mathbb{R}^k$ is onto, or, equivalently, if the matrix of the differential $[Df_p] = [\frac{\partial F_i}{\partial x_j}(p)]$ has rank k.*

Remark 0.0.5. If c is a regular value of $F\colon U \to \mathbb{R}^k$ where U is an open subset of $\mathbb{R}^n$ and the level set $\{F = c\}$ is non-empty then $n \geqslant k$.

Example 0.0.6. Let c is a regular value of $F\colon U \to \mathbb{R}^k$ where U is an open subset of $\mathbb{R}^{n+k}$.

Then $M = \{F = c\}$ has a natural structure of a generalized manifold of dimension n defined as follows.

Let $p \in M$. Since c is a regular value the matrix of partial derivatives $[Df_p] = [\frac{\partial F_i}{\partial x_j}(p)]$ has rank k. Therefore, we can find k linearly independent columns in $[Df_p]$.

By possibly renumbering the coordinates in $\mathbb{R}^{n+k}$ we can assume that the last k columns are linearly independent. let $x = (x_1, \ldots, x_n)$ be the first n coordinates in $\mathbb{R}^{n+k}$ and let $y = (y_1, \ldots, y_k)$ be the last k coordinates. Then the implicit function theorem say that there exists an open set W_p in $\mathbb{R}^{n+k}$ containing p such that $W_p \cap \{F = c\}$ is equal to the graph of a *smooth* function $y = y(x)$ mapping $V \to \mathbb{R}^k$ where $V \subset \mathbb{R}^n$ is open.

This defines a parameterization $\psi_p\colon V_p \to U_p = W \cap \{F = c\}$ given by $x \mapsto (x, y(x))$. Note that the inverse map $U_p \to V_p$ is given by $\varphi_p(x, y) = x$ and is a restriction to U_p of a smooth map defined on all of $\mathbb{R}^{n+k}$ by the same formula $(x, y) \mapsto x$.

We claim that the collection $\{\psi_p\}_{p\in M}$ defines a smooth atlas on M. (note that this atlas is never countable if $M \neq \emptyset$).

To see this we need to check that for any $p, q \in M$ the sets $\psi_p^{-1}(U_q)$ and $\psi_q^{-1}(U_p)$ are open (Exercise) and the transition maps $\psi_q^{-1} \circ \psi_p$ and $\psi_p^{-1} \circ \psi_q$ are smooth where defined. To see the latter recall that ψ_p and ψ_q are smooth and ψ_q^{-1} and ψ_p^{-1} are just restrictions of the coordinate projection maps $\mathbb{R}^{n+k} \to \mathbb{R}^n$ which are smooth as well. Therefore the compositions $\psi_q^{-1} \circ \psi_p$ and $\psi_p^{-1} \circ \psi_q$ are smooth too.

One can also show that M is Hausdorff and admits a countable atlas (Homework).