1. Differential forms on smooth manifolds

Definition 1.1. Let M^n be a smooth manifold (possibly with boundary). Define the tensor bundle of k-tensors over M as the disjoint union $\mathcal{T}^k(M) := \bigsqcup_{p\in M} \mathcal{T}^k(T_pM)$. Similarly, the bundle of alternating k-tensors on M is defined as $\mathcal{A}^k(M) := \bigsqcup_{p\in M} \mathcal{A}^k(T_pM)$.

We will mostly consider $\mathcal{A}^k(M)$, however, all the following constructions work for $\mathcal{T}^k(M)$ too.

We are going to show that $\mathcal{A}^k(M)$ admits a natural structure of a smooth manifold. The construction is completely analogous to the construction of the smooth structure on TM. However, at the moment it's just a family of vector spaces with no topology or smooth structure.

Remark 1.2. In the notations of the book $\mathcal{A}^k(M) = \Lambda^k(T^*M)$

We have a canonical projection $\pi\colon \mathcal{A}^k(M) \to M$ given by $\pi(p, w) = p$ where $p \in M$ and $w \in \mathcal{A}^k(T_pM)$. The fiber $\pi^{-1}(p)$ is equal to $\mathcal{A}^k(T_pM)$.

Given a diffeomorphism $f\colon M \to N$ its differential $df_p\colon T_pM \to T_{f(p)}N$ is a linear isomorphism for every $p \in M$. It induces a pullback map $df_p^*\colon \mathcal{A}^k(T_{f(p)}N) \to \mathcal{A}^k(T_pM)$ which is also a linear isomorphism. Since f is a bijection we can define

$$df^*\colon \mathcal{A}^k(N) \to \mathcal{A}^k(M)$$

as follows. For $q \in N, w \in \mathcal{A}^k(T_qN)$ set

$$df^*(q, w) := df^*_{f^{-1}(q)}(w)$$

It's easy to see that df^* is a bijection which is a linear isomorphism on all fibers.

Let $f\colon M \to N$, $g\colon N \to P$ be diffeomorphisms. Then $d(g \circ f) = dg \circ df$ and hence

$$d(g \circ f)^* = df^* \circ dg^*$$

Next let's consider the case $M = V$ is an open subset of $\mathbb{R}^n$.

Then for any $p \in M$ the tangent space T_pM has a canonical basis $e_1 = \frac{\partial}{\partial x_1}|_p, \ldots, e_n = \frac{\partial}{\partial x_n}|_p$. Therefore T_p^*M has a canonical basis $e^1, \ldots, e^n$ were $e^i(e_j) = \delta_{ij}$.

Let $x^i\colon \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate map. Consider $(dx^i)_p\colon T_p\mathbb{R}^n \to \mathbb{R}$. Then $(dx^i)_p(\frac{\partial}{\partial x_j}|_p) = \frac{\partial x^i}{\partial x_j}|_p = \delta_{ij}$. In other words,

$$(dx^i)_p = e^i \qquad i = 1, \ldots, n$$

From now on we will use the notations $(dx_i)_p$ for the elements of the dual basis instead of e^i. However, these are the same objects and this is simply a notation change.

Similarly, instead of writing $e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$ we will write $dx^I|_p = dx^{i_1}|_p \wedge \ldots \wedge dx^{i_k}|_p$.

Every element $w \in \mathcal{A}^k(T_pM)$ has a unique representation in the form $w = \sum_{I=(i_1<\ldots<i_k)} w_I dx^I|_p$. This gives a canonical bijection

$V \times \mathbb{R}^{\binom{n}{k}} \to \mathcal{A}^k(V)$ given by $(p, \{w_I\}) \mapsto \sum_I w_I dx^I|_p$ with the inverse map given by

$$(p, w) \mapsto (p, \{w(e_I)\})$$

Since both these maps are canonical we will identify $\mathcal{A}^k(V)$ with $V \times \mathbb{R}^{\binom{n}{k}}$ without explicitly writing these maps.

Now let M^n be a smooth manifold (possibly with boundary). Let $\{\phi_\alpha \colon V_\alpha \to U_\alpha\}_{\alpha\in\mathcal{A}}$ be an atlas on M where U_α is open in M and V_α is open in $\mathbb{R}^n$ (in $\mathbb{H}^n$ if M has boundary). Let $x_\alpha = \phi_\alpha^{-1} \colon U_\alpha \to V_\alpha$ be the corresponding local coordinate maps.

Since each x_α is a diffeomorphism, by above $x_\alpha^* \colon \mathcal{A}^k(V) = V \times \mathbb{R}^{\binom{n}{k}} \to \mathcal{A}^k(U)$ is a bijection for each α. Also note that $V \times \mathbb{R}^{\binom{n}{k}} \subset \mathbb{R}^{n+\binom{n}{k}}$ is open.

Lemma 1.3. *The collection $\{\Psi_\alpha = x_\alpha^*\}_{\alpha\in\mathcal{A}}$ defines a smooth atlas on $\mathcal{A}^k(M)$.*

Proof. The main thing is to check that the transition maps are smooth. Let $\alpha, \beta \in \mathcal{A}$ be arbitrary and consider the map $\Psi_{\alpha\beta} = \Psi_\beta^{-1} \circ \Psi_\alpha$. Let $f = \phi_\beta^{-1} \circ \phi_\alpha \colon V_{\alpha\beta} \to V_{\beta\alpha}$. Let's denote the coordinates on the the domain by $x = (x_1, \ldots, x_n)$ and on the target by $y = (y_1, \ldots, y_n)$. Then

$$\Psi_{\alpha\beta}(x, \{w_I\}) = (f(x), \{w'_J\})$$

Let us determine the dependence of w'_J on $x, \{w_I\}$. Let $p = \phi_\alpha(x)$. We have that $\sum_I w_I dx^I|_p = \sum_J w'_J dy^J|_p$. Fix a multi-index $J = (j_1, \ldots, j_k)$ and evaluate this equality on $\frac{\partial}{\partial y_{j_1}}|_p, \ldots, \frac{\partial}{\partial y_{j_k}}|_p$. We have $RHS = w'_J$. $LHS = \sum_I w_I dx^I|_p(\frac{\partial}{\partial y_{j_1}}|_p, \ldots, \frac{\partial}{\partial y_{j_k}}|_p)$. We have $\frac{\partial}{\partial y_j}|_p = \sum_i \frac{\partial x_i}{\partial y_j}\frac{\partial}{\partial x_i}|_p$.

This gives

$$w_J = \sum_I w_I dx^I|_p(\frac{\partial}{\partial y_{j_1}}|_p, \ldots, \frac{\partial}{\partial y_{j_k}}|_p) = \sum_I w_I dx^I(\sum_{q_1} \frac{\partial x_{q_1}}{\partial y_{j_1}}\frac{\partial}{\partial x_{q_1}}|_p, \ldots, \sum_{q_1} \frac{\partial x_{q_k}}{\partial y_{j_k}}\frac{\partial}{\partial x_{q_k}}|_p)$$

$$= \sum_I \sum_{Q=(q_1,\ldots,q_k)} w_I \frac{\partial x_{q_1}}{\partial y_{j_1}} \cdot \ldots \cdot \frac{\partial x_{q_k}}{\partial y_{j_k}} dx^I(\frac{\partial}{\partial x_{q_1}}|_p, \ldots, \frac{\partial}{\partial x_{q_k}}|_p)$$

is smooth since $y = y(x)$ and $x = x(y)$ are smooth. □

Corollary 1.4. *The canonical projection $\pi \colon \mathcal{A}^k(M) \to M$ is a submersion.*

Remark 1.5. Note that by construction the coordinate maps Ψ_α commute with π and are linear isomorphisms on the fibers of π.

Definition 1.6. A smooth differential k-form ω on a manifold M^n is a smooth section of the bundle $\pi \colon \mathcal{A}^k(M) \to M$, i.e. it's a smooth map

$$\omega \colon M \to \mathcal{A}^k(M)$$

such that $\pi \circ \omega = \mathrm{id}_M$, i.e. $\omega(p) \in \mathcal{A}^k(T_pM)$ for any $p \in M$.

Given local coordinates $x\colon U \to V$ $U \subset M, V \subset \mathbb{R}^n$ are open, $\omega|_U$ can be written as $\sum_I \omega_I(x) dx^I$. The following Lemma is an easy consequence of the definition

Lemma 1.7. *Let $\omega\colon M \to \mathcal{A}^k(M) \to M$ be a map. TFAE*

(1) ω is smooth

(2) For any local coordinate chart $x\colon U \to V$ where $U \subset M, V \subset \mathbb{R}^n$ are open $\omega|_U = \sum_I \omega_I(x) dx^I$ and all $w_I(x)$ are smooth functions.

(3) There exists an atlas $\{\phi_\alpha\colon V_\alpha \to U_\alpha\}_{\alpha\in\mathcal{A}}$ be an atlas on M such that for any α we have

$$\omega|_{U_\alpha} = \sum_I \omega_I^\alpha(x_\alpha) dx_\alpha^I$$

and all ω_I^α are smooth.

We denote the set of all smooth k-form on M by $\Omega^k(M)$. We'll denote by $\Omega^*(M)$ the collection of all forms of all degrees i.e. $\cup_k \Omega^k(M)$.

Note that $\Omega^0(M) = C^\infty(M)$. All pointwise operations on alternating tensors such as addition, multiplication by a number and wedge product make sense for forms Moreover, if $\omega \in \Omega^k(M)$ and $f\colon M \to \mathbb{R}$ is smooth then $f \cdot \omega$ is also a smooth form.

Pullbacks make sense for forms as well.

Given a smooth map $f\colon M \to N$ and $\omega \in \Omega^k(N)$ we define $f^*(\omega)$ by $f^*(\omega)(p) = df_p^*(\omega(f(p)))$. I.e. for $v_1, \dots v_k \in T_pM$ we have $f^*\omega(p)(v_1, \dots, v_k) = \omega(f(p))(df_p(v_1), \dots, df_p(v_k))$. By computing $f^*(\omega)$ in local coordinates it follows from Lemma 1.7 that $f^*(\omega)$ is smooth.

Proposition 1.8.

a) If $\omega_1, \omega_2 \in \Omega^k(M)$, $f_1, f_2\colon M \to \mathbb{R}$ are smooth then $f_1\omega_1 + f_2\omega_2 \in \Omega^k(M)$ is also a smooth k-form.

b) If $\omega, \eta \in \Omega^*(M)$ then $\omega \wedge \eta \in \Omega^*(M)$

c) If $F\colon M \to N$ is smooth then $F^*\colon \Omega^*(N) \to \Omega^*(N)$ is linear. Moreover $F^*(g) = g \circ F$ for any $g \in \Omega^0(N)$.

d) If $F\colon M \to N$ and $G\colon N \to P$ are smooth and then $(G \circ F)^* = F^* \circ G^*$

e) If $f\colon M \to \mathbb{R}$ is smooth then $f^*(dt) = df$ - differential of f which in local coordinates x on M can be written as $\sum_i \frac{\partial f}{\partial x_i} dx^i$

f) If $F\colon M \to N$ is smooth and $\omega, \eta \in \Omega^*(N)$ then $F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$

g) If $F = (F_1, \dots, F_m)\colon M \to \mathbb{R}^m$ is smooth then $F^*(\sum_{I=(i_1<\dots<i<k)} w_I(y) dy^I) = \sum_I (\omega_I \circ F) dF_{i_1} \wedge \dots \wedge dF_{i_k}$

Proof. a),b),c), f) are straightforward. d) follows from the definition of pullback and the chain rule $d(g \circ f) = dg \circ df$. e) is immediate from the definition: for $p \in M, v \in T_p(M)$ we have $f^*(dt)(v) = dt(df_p(v)) = df_p(v)$.

To get the coordinate expression for df recall that for any 1-form ω we have $\omega = \sum_i \omega(\frac{\partial}{\partial x_i}) dx^i$. In case of $\omega = df$ this gives $df = \sum_i df(\frac{\partial}{\partial x_i}) dx^i = \sum_i \frac{\partial f}{\partial x_i} dx^i$

g) follows from e), f):

$$F^*(\sum_{I=(i_1<\ldots<i<k)} w_I(y)dy^I) = \sum_I F^*(w_I(y)dy^{i_1}\wedge\ldots\wedge dy^{i_k}) = \sum_I(\omega_I\circ F)F^*(dy^{i_1})\wedge\ldots\wedge F^*(dy^{i_k}) =$$

$$\sum_I (\omega_I \circ F)dF^{i_1} \wedge \ldots \wedge dF^{i_k}$$

□

Formula g) from the previous Proposition has a particularly simple form for top dimensional forms:

Lemma 1.9. *Let $F = (F_1, \ldots, F_n)\colon U \to V$ be smooth where $U, V \subset \mathbb{R}^n$ are open. Let $\omega = u(y)dy^1 \wedge \ldots dy^n$ be an n-form on V. Then*

$$F^*(\omega) = (u(F(x))(\det(\frac{\partial f_i}{\partial x_j}))dx^1 \wedge \ldots \wedge dx^n$$

2. Exterior derivative

Definition 2.1. Let $\omega = \sum_I \omega_I(x)dx^I$ be a smooth differential form on an open subset V in $\mathbb{R}^n$. Define the *exterior derivative* $d\omega$ by the formula

$$d\omega = \sum_I d\omega_I(x) \wedge dx^I$$

Proposition 2.2.

a) Let $f\colon V \to \mathbb{R}$ be smooth. Then $df = \sum_i \frac{\partial f}{\partial x_i}dx^i$ when viewed as exterior derivative of f an a 0-form coincides with df - differential of f.

b) $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is linear

c) $d \circ d = 0$

d) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|\cdot|\eta|}\omega \wedge d\eta$

e) If $F\colon V_1 \to V_2$ is smooth where $V_1 \subset \mathbb{R}^n, V_2 \subset \mathbb{R}^m$ are open (m and n need not be equal) and ω is a form on V_2 then

$$F^*(d\omega) = d(F^*(\omega))$$

Proof. a) is proved above in the proof of Proposition 1.8. e) is a formal consequence of b),c), d) and Proposition 1.8:

By Proposition 1.8 we know that if $\omega = dg$ where $g\colon N \to R$ is smooth then $F^*(dg) = d(g \circ F))$.

By linearity it's enough to prove e) for $\omega = u(y)dy^I$. Then $d\omega = du \wedge dy^I$. Hence $F^*(d\omega) = F^*(du \wedge dy^I) = F^*(du) \wedge F^*(dy^I) = d(u \circ F) \wedge dF^{i_1} \wedge \ldots \wedge dF^{i_k}$.

One the other hand, $F^*(\omega) = (u \circ F)dF^{i_1} \wedge \ldots \wedge dF^{i_k}$. Then by repeatedly applying d) and using that $d(dF_i) = 0$ we get that

$$dF^*(\omega) = d(u \circ F) \wedge dF^{i_1} \wedge \ldots \wedge dF^{i_k}$$

□

Definition 2.3. Let M^n be a smooth manifold and let ω be a smooth k-form on M. Define $d\omega \in \Omega^{k+1}(M)$ as follows. Let $x\colon U \to V$ be a local coordinate chart where $U \subset M, V \subset \mathbb{R}^n$ are open.

Define $d\omega$ on U as follows. Let $\alpha = (x^{-1})^*\omega$ be ω in x coordinates. Then set

$$d\omega := x^*(d\alpha)$$

Lemma 2.4. *$d\omega$ is well defined, i.e. it does not depend on the choice of a coordinate chart x.*

Proof. Let $x\colon U_1 \to V_1, y\colon U_2 \to V_2$ be two different charts.

Observe that two forms α on V_1, and β on V_2 define the same form on $U_1 \cap U_2$. i.e. $x^*(\alpha) = y^*(\beta)$ iff $\beta = (y^{-1})^*(y^*(\beta)) = (y^{-1})^*(x^*(\alpha)) = (x \circ y^{-1})^*(\alpha) = F^*(\alpha)$ where F is the transition map $x \circ y^{-1}$

Let $\alpha = x^*(\omega)$,$\beta = y^*(\omega)$. By above $\beta = F^*(\alpha)$ and to show that $d\omega$ is well defined we need to check that $F^*(d\alpha) = d\beta$. But this is true by Proposition 2.2e) because d commutes with pullbacks. $\square$