3. Exterior derivative (continued)

Proposition 3.1. *Exterior differentiation on a manifold M satisfies the following properties*

a) Let $f\colon M \to \mathbb{R}$ be smooth. Then df when viewed as exterior derivative of f as a 0-form coincides with df- differential of f.
b) $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is linear.
c) $d \circ d = 0$
d) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|\cdot|\eta|}\omega \wedge d\eta$
e) If $F\colon M^n \to N^m$ is smooth and ω is a form on N then

$$F^*(d\omega) = d(F^*(\omega))$$

Moreover, an operation $\mathsf{d}\colon \Omega^*(M) \to \Omega^{*+1}(M)$ satisfying a)-d) is unique and must coincide with the exterior derivative.

Proof. The proof of the properties a)-d) is an immediate consequence of the definition and the fact that these properties hold for exterior derivatives on open subsets of $\mathbb{R}^n$.

To prove uniqueness, suppose d is another operation satisfying a)-d). Observe that locally any ω can be written as $\omega = \sum_I \omega_I(x)dx^I$ where $dx^I = dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $x\colon U \to V \subset \mathbb{R}^n$ is a local coordinate chart on an open subset $U \subset M$. Then we have that $\mathsf{d}x^i = dx^i$ by a) and hence $\mathsf{d}(dx^i) = \mathsf{d}(\mathsf{d}x^i) = 0$ by c). Therefore, by repeatedly applying d) we get that $\mathsf{d}(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) = 0$. Therefore, by d) again we get that $\mathsf{d}(\omega_I(x)dx^I) = \mathsf{d}(\omega_i(x)) \wedge dx^I + \omega_I(x) \wedge \mathsf{d}(dx^I) = dw_i(x) \wedge dx^I + 0 = d(\omega_I(x)dx^I)$. The general case follows by linearity of d and d. $\square$

4. De Rham cohomology

Definition 4.1. A form $\omega \in \Omega^*(M)$ is called *closed* if $d\omega = 0$.

A form $\omega \in \Omega^*(M)$ is called *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{*-1}(M)$.

Since $d \circ d = 0$ it's obvious that every exact form is closed. It's natural to ask to what extent the converse holds. Let $B^k(M)$ be the set of all exact k-forms and let $Z^k(M)$ be the set of all closed k forms. It's obvious that $B^k(M), Z^k(M)$ are vector spaces and by above $B^k(M) \subset Z^k(M)$.

Definition 4.2. Let M^n be a smooth manifold, possibly with boundary. The k-th de Rham cohomology group of M is defined to be the quotient group

$$H^k_{DR}(M) := Z^k(M)/B^k(M)$$

Since $B^k(M)$ is a vector subspace of $Z^k(M)$ the quotient $H^k_{DR}(M)$ is a vector space and not just a group.

By the definition that $H^k_{DR}(M) = 0$ iff every closed k-form is exact.

Example 4.3. Let $M = V$ be an open subset of $\mathbb{R}^2$. Then a 1-form ω on V has the form $P(x,y)dx + Q(x,y)dy$. By definition, w is exact iff $\omega = df$ for some smooth $f\colon V \to \mathbb{R}$, i.e. if $P(x,y)dx + Q(x,y)dy = \frac{\partial f}{\partial x}(x,y)dx + \frac{\partial f}{\partial y}(x,y)dxy$, or $P(x,y) = \frac{\partial f}{\partial x}(x,y)$ and $Q(x,y) = \frac{\partial f}{\partial y}(x,y)$.

On the other hand ω is closed iff $0 = d\omega = d(P(x,y)dx + Q(x,y)dy) = (-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y))dx \wedge dy$ or $-\frac{\partial P}{\partial y}(x,y) + \frac{\partial Q}{\partial x}(x,y) = 0$.

Thus, every closed 1-form on V is exact iff for any smooth $P, Q\colon V \to \mathbb{R}$ satisfying $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y)$ there exists a smooth $f\colon V \to \mathbb{R}$ such that $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}(x,y)$.

Exercise 4.4. *Prove that* $H^1_{DR}(\mathbb{R}^2) = 0$

Let $f\colon M \to N$ be a smooth map between manifolds. Since F^* commutes with d, F^* sends closed forms to closed forms and exact forms to exact forms. Therefore it induces a homomorphism $F^*\colon H^k_{DR}(N) \to H^k_{DR}(M)$ for any k.

Since $(G \circ F)* = F^* \circ G^*$ and $Id^*_M = Id$ it follow that if $F\colon M \to N$ is a diffeomorphism then $F^*\colon H^k_{DR}(N) \to H^k_{DR}(M)$ is an isomorphism. We will see later that for $V = \mathbb{R}^2 \backslash \{0\}$ the form $\omega = \frac{y}{x^2+y^2}dx - \frac{x}{x^2+y^2}dy$ is closed but not exact. This will imply that $H^1_{DR}(\mathbb{R}^2 \backslash \{0\}) \neq 0$. Since $H^1(\mathbb{R}^2) = 0$ by the exercise above, this will show that $\mathbb{R}^2$ is not diffeomorphic to $\mathbb{R}^2 \backslash \{0\}$.

5. Orientation

5.1. **Orientation on a vector space.**

Definition 5.1. Let V be a finite dimensional vector space. Let $e = (e_1, \dots, e_n)$ and $e' = (e'_1, \dots, e'_n)$ be two bases of V. We say that $e \sim e'$ if the transition matrix A from e to e' has $\det A > 0$. It's easy to see that $\sim$ satisfies the following properties

- if $e \sim e'$ then $e' \sim e$;
- if $e \sim e'$ and $e' \sim e''$ then $e \sim e''$.

This means that $\sim$ is an equivalence relation on the set of all bases of V. We will call equivalence classes mod $\sim$ *orientations* on V. We will say that two bases e, e' *have the same orientation* if they belong to the same equivalence class, i.e. the transition matrix from e to e' has positive determinant.

Lemma 5.2. *Let V be a finite dimensional vector space. Then there are precisely two possible ordinations on V.*

Proof. Let $e = (e_1, \dots, e_n)$ be a basis of V and let $e' = (-e_1, e_2 \dots, e_n)$. Since the transition matrix A from e to e' has determinant -1 they define two different orientations on V. We claim that any other basis of V is equivalent to either e or e': Let e'' be a basis of V. Let B be the transition matrix from e' to e''. Then the transition matrix from e to e'' is BA and $\det(BA) = \det B {\cdot} \det A = -\det B$. This means that $\det B$ and $\det(BA)$ have opposite signs, and thus one of them is positive and the other is negative. Therefore $e'' \sim e$ or $e'' \sim e'$. $\square$

We'll call the two distinct orientations on V *opposite or negative* to each other. If ϵ is an orientation and $e = (e_1, \dots, e_n)$ is a basis we put $\epsilon(e) = +1$ if e is positively oriented with respect to ϵ and we put $\epsilon(e) = -1$ if e is negatively oriented with respect to ϵ.

$\mathbb{R}^n$ has a canonical orientation defined by the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$.

Orientations on V correspond to orientations on $\mathcal{A}^n(V) \cong \mathbb{R}$ as follows.

Let $w \in \mathcal{A}^n(V)$ be a nonzero alternating n-tensor. It defines an orientation ϵ_w as follows:

Given a basis $e = (e_1, \dots, e_n)$ we'll say that e is positively oriented iff $w(e_1, \dots, e_n) > 0$. It's easy to see that this defines an orientation on V. It's also obvious that if $w' = \lambda w$ with $\lambda \neq 0$ then w and w' define the same orientation iff $\lambda > 0$.

5.2. **Orientation on manifolds.** Let M^n be a smooth n-dimensional manifold (possibly with boundary)

Definition 5.3. An orientation ϵ on M^n is a choice of orientation $\epsilon(p)$ on T_pM for all $p \in M$.

An orientation ϵ is called *continuous* if for any $p \in M$ there exists an open set $U \subset M$ containing p and a collection of continuous vector fields $X_1, \dots X_n$ on U such that $X_1(q), \dots X_n(q)$ is a basis of T_qM for any $q \in U$ and $\epsilon(X_1(q), \dots X_n(q)) = +1$ for any $q \in U$.

A manifold M is called *orientable* if it admits a continuous orientation.

Exercise 5.4. *Prove that an orientation ϵ is continuous if and only if it's smooth, i.e. for any $p \in M$ there exists an open set $U \subset M$ containing p and a collection of* **smooth** *vector fields $X_1, \dots X_n$ on U such that $X_1(q), \dots X_n(q)$ is a basis of T_qM for any $q \in U$ and $\epsilon(X_1(q), \dots X_n(q)) = +1$ for any $q \in U$.*

From now on we will only consider continuous orientations. The relation between orientations and nonzero alternating n-vectors on a fixed vector space naturally carries over to manifolds as follows.

Suppose ω is a smooth n-form on M^n such that $\omega(p) \neq 0$ for any $p \in M$. Then ω defines an orientation ϵ_ω on M as follows. Given $p \in M$ and a basis $v_1, \dots v_n$ of T_pM we say that it's positively oriented iff $\omega(p)(v_1, \dots v_n) > 0$.

Lemma 5.5. *ϵ_ω is continuous.*

Proof. Let $p \in M$ be any point. Let U be a coordinate ball containing p, so U is diffeomorphic to an open ball $B(0,1)$ in $\mathbb{R}^n$ under some local coordinate map $x\colon U \to \mathbb{R}^n$. Let $X_i(q) = \frac{\partial}{\partial x_i}(q)$. Then $f(q) = \omega(X_1(q), \dots, X_n(q))$ is smooth on U. Since $f(q) \neq 0$ for any q, by the Intermediate Value Theorem we must have that $f(q) > 0$ for all $q \in U$ or $f(q) < 0$ for all $q \in U$. In the first case this gives the required collection of continuous vector fields on U. In the second case the same works after changing X_1 to $-X_1$. $\square$

Next we will show that the converse also holds, i.e. every continuous orientation is equal to ϵ_ω for some nowhere zero $\omega \in \Omega^n(M^n)$.

Lemma 5.6. *Let ϵ be a continuous orientation on a smooth manifold M^n. Then there exists a smooth form $\omega \in \Omega^n(M)$ such that $\omega(p) \neq 0$ for any $p \in M$ and $\epsilon = \epsilon_\omega$.*

Proof. Let ϵ be a continuous orientation on M.

We will use the following terminology. Let ω be a smooth n-form on an open subset $U \subset M$. We will say that ω is *positive* on U if $\omega(p)(v_1, \dots, v_n) > 0$ for any $p \in U$ and any positive basis $v_1, \dots, v_n$ of T_pM. We need to prove that there exists a positive form on $U = M$.

Observe that if $\omega_1, \dots, \omega_m$ are positive forms on U and $\phi_1, \dots \phi_m \colon U \to \mathbb{R}$ are smooth functions such that $\phi_i \geq 0$ on U and $\sum_i \phi_i > 0$ on U then $\sum_i \phi_i \omega_i$ is positive on U.

For any $p \in M$ let U_p be an open set containing p such that there exist n smooth vector fields $X_1, \dots X_n$ on U_p such that $X_1(q), \dots X_n(q)$ is a positive basis of T_qM for any $q \in U_p$. Let $X^1(q), \dots, X^n(q)$ be the dual basis of T_q^*M. Then $X^1, \dots, X^n$ are smooth forms on U (why?). Let $\omega_p = X^1 \wedge \dots \wedge X^n$. Then it's a smooth positive form on U. Take a partition of unity $\{\phi_i\}$ subordinate to the cover $\{U_p\}_{p \in M}$ of M. Then $\operatorname{supp} \phi_i \subset U_{p_i}$ for some p_i and by the observation above $\omega = \sum_i \phi_i \omega_i$ is positive on all of M. □