1. Tensors on vector spaces

Let V be a finite dimensional vector space over $\mathbb{R}$.

Definition 1.1. A tensor of type (k, l) on V is a map

$$T\colon \underbrace{V \times \ldots \times V}_{k \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}} \to \mathbb{R}$$

which is linear in every variable.

Example 1.2.

- Let $v \in V$ be a vector. Then v defines a tensor T_v of type $(0, 1)$ with the map $T_v\colon V^* \to \mathbb{R}$ given by $T_v(f) = f(v)$. The map $v \mapsto T_v$ gives a linear isomorphism from V onto $(V^*)^*$ = space of all tensors of type $(0, 1)$.
- Let $\langle \cdot, \cdot \rangle$ be an inner product on V. Then it is a tensor of type $(2, 0)$.
- Let $V = \mathbb{R}^n$ and let $T\colon \underbrace{V \times \ldots \times V}_{n \text{ times}} \to \mathbb{R}$ be given by $T(v_1, \ldots, v_n) = \det A$ where A is the $n \times n$ matrix with columns $v_1, \ldots, v_n$. Then T is a tensor of type $(n, 0)$.

From now on we will only consider tensors of type $(k, 0)$ which we'll refer to as simply *k-tensors*. Let $\mathcal{T}^k(V)$ be the set of all k tensors on V. It's obvious that $\mathcal{T}^k(V)$ is a vector space and $\mathcal{T}^1(V) = V^*$. Also $\mathcal{T}^0(V) = \mathbb{R}$.

Definition 1.3. Let V, W be vector spaces and let $L\colon V \to W$ be a linear map. Let T be a k-tensor on W. Let $L^*(T)\colon \underbrace{V \times \ldots \times V}_{k \text{ times}} \to \mathbb{R}$ be defined by the formula

$$L^*(T)(v_1, \ldots, v_k) := T(L(v_1), \ldots, L(v_k))$$

Then it's immediate that $L^*(T)$ is a k-tensor on V which we'll call *the pullback of T by L.*

It's easy to see that $L^*\colon \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ is linear.

Definition 1.4. Let $T \in \mathcal{T}^k(V)$, $S \in \mathcal{T}^l(V)$. We define their *tensor product* $T \otimes S \in \mathcal{T}^{k+l}(V)$ by the formula

$$T \otimes S(v_1, \ldots, v_{k+l}) = T(v_1, \ldots, v_k) \cdot S(v_{k+1}, \ldots, v_{k+l})$$

It's obvious that $T \otimes S$ is a tensor. The following properties of tensor product are obvious from the definition

- Tensor product is associative: $(T \otimes S) \otimes R = T \otimes (S \otimes R)$
- tensor product is linear in both variables: $(\lambda_1 T_1 + \lambda_2 T_2) \otimes R = \lambda_1 T_1 \otimes R + \lambda_2 T_2 \otimes R$ and the same holds for R.
- tensor product commutes with pullback, i.e. if $L\colon V \to W$ is a linear map between vector spaces and T, S are tensors on W then

$$L^*(T \otimes S) = L^*(T) \otimes L^*(S)$$

Let us construct a basis of $\mathcal{T}^k(V)$ and compute its dimension. Let $e_1, \dots, e_n$ be a basis of V and let $e^1, \dots e^n$ be the dual basis of V^*, i.e.

$$e^i(e_j) = \delta_{ij}$$

For any multi-index $I = (i_1, \dots, i_k)$ with $1 \le i_j \le n$ define ϕ^I as $\phi^I = e^{i_1} \otimes \dots \otimes e^{i_k}$. Also, we will denote the k-tuple $(e_{i_1}, \dots e_{i_k})$ by e_I.

It's immediate from the definition that

$$\phi^I(e_J) = \delta_{IJ} = \begin{cases} 1 \text{ if } I = J \\ 0 \text{ if } I \neq J \end{cases} \tag{1}$$

For example $e^1 \otimes e^2(e_2, e_1) = e^1(e_2) \cdot e^2(e_1) = 0$.

Lemma 1.5. *Let $T, S \in \mathcal{T}^k(V)$ then $T = S$ iff $T(e_I) = S(e_I)$ for any multi-index $I = (i_1, \dots i_k)$.*

Proof. This follows immediately from multi-linearity of T and S. □

Lemma 1.6. *The set $\{\phi^I\}_{I=(i_1,\dots,i_k)}$ is a basis of $\mathcal{T}^k(V)$. In particular,* $\dim \mathcal{T}^k(V) = n^k$

Proof. Let us first check linear independence of ϕ^I's. Suppose $\sum_I \lambda_I \phi^I = 0$. Let $J = (j_1, \dots, j_k)$ be a multi-index of length k. Using (1) we obtain

$$0 = (\sum_I \lambda_I \phi^I)(e_J) = \sum_I \lambda_I \phi^I(e_J) = \sum_I \lambda_I \delta_{IJ} = \lambda_J$$

Since J was arbitrary this proves linear independence of $\{\phi^I\}_{I=(i_1,\dots,i_k)}$.

Let us show that they span $\mathcal{T}^k(V)$.

Let $T \in \mathcal{T}^k(V)$. Let $S = \sum_I T(e_I)\phi^I$. Then S belongs to the span of $\{\phi^I\}_I$. For any J we have that $S(e_J) = \sum_I T(e_I)\phi^I(e_J) = \sum_I T(e_I)\delta_{IJ} = T(e_J)$. Therefore, $T = S$ by Lemma 1.5 and hence T belongs to the span of $\{\phi^I\}_I$.

□

2. Alternating tensors

Definition 2.1. Let V be a finite dimensional vector space. A k-tensor T on V is called alternating if for any $v_1, \dots, v_k$ and any $1 \le i < j \le k$ we have

$$T(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -T(v_1, \dots, v_j, \dots, v_i, \dots, v_k)$$

Example 2.2.

- Any 1-tensor on V is alternating.
- The determinant tensor defined in Example 1.2 is alternating.
- More generally, let $e_1, \dots e_n$ be a basis of V. Let $I = (i_1, \dots, i_k)$ be a k-multi-index. Define a k tensor e^I as follows.

Let $v_1, \dots, v_k \in V$. Let A be the $n \times k$ matrix whose i-th column is given by the coordinates of v_i with respect to the basis $e_1, \dots e_n$. Let A^I be the $k \times k$ matrix made of rows $i_1, \dots, i_k$ of A.

Define e^I by the formula

$$e^I(v_1, \dots, v_k) = \det A^I$$

It's immediate that e^I is alternating because the determinant of a matrix changes sign if two of its columns are switched. It's also obvious that if I has some repeating indices then $e^I = 0$.

The following properties of alternating tensors are immediate from the definition

- Let $\mathcal{A}^k(V)$ be the set of all alternating k tensors on V. Then $\mathcal{A}^k(V)$ is a vector subspace of $\mathcal{T}^k(V)$, i.e. a linear combination of alternating tensors is alternating.
- If $L\colon V \to W$ is linear and $\omega \in \mathcal{A}^k(W)$ then $L^*(\omega) \in \mathcal{A}^k(W)$

Remark 2.3. In the notations of the book $\mathcal{A}^k(V) = \Lambda^k(V^*)$.

Let $\sigma \in S_k$ be a permutation and let $T \in \mathcal{T}^k(A)$ be a k-tensor. Define ${}^\sigma T$ by

$${}^\sigma T(v_1, \dots, v_k) = T(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

The following properties are immediate from the definition

- ${}^\sigma(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 {}^\sigma T_1 + \lambda_2 {}^\sigma T_2$
- ${}^{\sigma\tau}T = {}^\sigma({}^\tau T)$

Lemma 2.4. *Let $T \in \mathcal{T}^k(V)$. Then TFAE*

(i) $T(v_1, \dots, v_k) = 0$ *if* $v_i = v_j$ *for some* $i \neq j$
(ii) T *is alternating;*
(iii) $T(v_1, \dots, v_k) = 0$ *if* $v_1, \dots, v_k$ *are linearly dependent.*
(iv) ${}^\sigma T = \operatorname{sign}\sigma \cdot T$ *for any* $\sigma \in S_k$.

Let $e_1, \dots e_n$ be a basis of V. Let $I = (i_1, \dots i_k), J = (j_1, \dots, j_k)$ be two multi-indices with $i_s \neq i_t, j_s \neq j_t$ for all $s \neq t$.

It's easy to see from the definition of e^I that $e^I(e_J) = 0$ if $\{i_1, \dots, i_k\} \neq \{j_1, \dots, j_k\}$ and $e^I(e_J) = \operatorname{sign}\sigma$ if $\{i_1, \dots, i_k\} = \{j_1, \dots, j_k\}$ and $\sigma \in S_k$ is the unique permutation satisfying $I = \sigma(J) = (j_{\sigma(1)}, \dots, j_{\sigma(k)})$

In particular, if $I = (i_1 < i_2 < \dots < i_k), J = (j_1 < j_2 < \dots < j_k)$ then

$$e^I(e_J) = \delta_{IJ} \tag{2}$$

Lemma 2.5. *Let $\alpha, \beta \in \mathcal{A}^k(V)$. Then $\alpha = \beta$ iff $\alpha(e_I) = \beta(e_I)$ for any $I = (i_1 < i_2 < \dots < i_k)$.*

Lemma 2.6. *The set $\{e^I\}_{I=(i_1<i_2<\dots<i_k)}$ is a basis of $\mathcal{A}^k(V)$. In particular $\dim \mathcal{A}^k(V) = \binom{n}{k}$ for $k \leq n$ and $\dim \mathcal{A}^k(V) = 0$ if $k > n$.*

Proof. The proof is the same as the proof of Lemma 1.6 but using Lemma 2.5 instead of Lemma 1.5 and (2) instead of (1). $\square$

Example 2.7. Any element of $\mathcal{A}^k(V)$ can be written as a linear combination of the standard basis $\{\phi^I\}$ of $\mathcal{T}^k(V)$. For example, if $n = \dim V = 4$ then $e^{13} = e^1 \otimes e^3 - e^3 \otimes e^1 = \phi^{13} - \phi^{31}$.

Lemma 2.8. *Let $L\colon V \to V$ be a linear map and let $A = [L]$ be the matrix of L with respect to the basis $e_1, \ldots, e_n$ of V. Then for any $w \in \mathcal{A}^n(V)$ we have that $L^*(w) = (\det A)w$.*

Proof. By Lemma 2.6, $\mathcal{A}^n(V)$ is 1-dimensional with basis given by $e^{12\ldots n}$. Therefore, it's enough to prove the lemma for $\omega = e^{12\ldots n}$.

We have $L^*(^{12\ldots n}) = \lambda^{12\ldots n}$ where $\lambda = L^*(^{12\ldots n})(e_1, \ldots, e_n) = e^{12\ldots n}(L(e_1), L(e_2), \ldots, L(e_n)) = \det A$ by definition of $e^{12\ldots n}$ and because columns of A are given by $L(e_1), L(e_2), \ldots, L(e_n)$ written in the basis $(e_1, \ldots, e_n)$. □

3. Wedge product

Let $T \in \mathcal{T}^k(V)$. Define $\operatorname{Alt}(T)$ as

$$\operatorname{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign} \sigma \cdot {}^{\sigma}T$$

Lemma 3.1.

a) $\operatorname{Alt}(T)$ *is alternating for any T.*
b) If $\omega \in \mathcal{A}^k(V)$ is alternating then $\operatorname{Alt}(\omega) = \omega$
c) $\operatorname{Alt}\colon \mathcal{T}^k(V) \to \mathcal{A}^k(V)$ *is linear.*
d) $\operatorname{Alt}({}^{\sigma}T) = \operatorname{sign} \sigma \cdot {}^{\sigma} \operatorname{Alt}(T)$
e) Alt *commutes with pullbacks: Let $L\colon V \to W$ be linear and let $T \in \mathcal{T}^k(W)$. Then*

$$\operatorname{Alt}(L^*(T)) = L^*(Alt(T))$$

Definition 3.2. Let $\omega \in \mathcal{A}^k(V), \eta \in \mathcal{A}^l(V)$. Then we define their *wedge product* $\omega \wedge \eta$ by the formula

$$\omega \wedge \eta = \frac{(k+l)!}{k! \cdot l!} \operatorname{Alt}(\omega \otimes \eta)$$

Lemma 3.3.

i) $e^I \wedge e^J = e^{IJ}$ *for any* $I = (i_1, \ldots, i_k), J = (j_1, \ldots j_l)$
ii) $(\lambda_1\omega_1 + \lambda_2\omega_2) \wedge \eta = \lambda_1\omega_1 \wedge \eta + \lambda_2\omega_2 \wedge \eta$
iii) $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$
iv) $\omega \wedge \eta = (-1)^{|\omega| \cdot |\eta|} \eta \wedge \omega$
v) Let $I = (i_1, \ldots, i_k)$ *then* $e^I = e^{i_1} \wedge \ldots \wedge e^{i_k}$
vi) For any $\omega_1, \ldots, \omega_k \in V^*, v_1, \ldots, v_k \in V$ *we have* $\omega_1 \wedge \ldots \wedge \omega_k(v_1, \ldots, v_k) = \det(\omega_i(v_j))$
vii) let $\omega_1, \ldots, \omega_k \in V^*$*. Then* $\omega_1 \wedge \ldots \wedge \omega_k = 0$ *iff* $\omega_1, \ldots, \omega_k$ *are linearly dependent.*

Proof. Parts i)-vi) are proved in the book (Propositions 14.10 and 14.11). Let's prove vii).

Note that for any $\omega \in V^*$ by part iv) we have that $\omega \wedge \omega = (-1)\omega \wedge \omega$ and hence $\omega \wedge \omega = 0$.

Suppose $\omega_1, \ldots, \omega_k$ are linearly dependent. Then we can write one of them as a linear combination of the others. WLOG we can assume $\omega_k = \sum_{i=1}^{k-1} \lambda_i \omega_i$. Therefore,

$$\omega_1 \wedge \ldots \wedge \omega_k = \omega_1 \wedge \ldots \wedge \omega_{k-1} \wedge (\sum_{i=1}^{k-1} \lambda_i \omega_i) = \sum_{i=1}^{k-1} \lambda_i \omega_1 \wedge \ldots \wedge \omega_{k-1} \wedge \omega_i = 0$$

because each summand contains a repeated factor.

Now suppose $\omega_1, \ldots, \omega_k$ are linearly independent. Then we can find $\omega_{k+1}, \ldots, \omega_n \in V^*$ such that $\omega_1, \ldots, \omega_n$ is a basis of V^*. Let $v_1, \ldots, v_n$ be the dual basis of V, i.e. $\omega_i(v_j) = \delta_{ij}$. Then by vi) we have $\omega_1 \wedge \ldots \wedge \omega_k(v_1, \ldots, v_k) = \det(\omega_i(v_j)) = 1$ and hence $\omega_1 \wedge \ldots \wedge \omega_k \neq 0$. □